J Stat Phys (2007) 129: 885–935
DOI 10.1007/s10955-007-9350-9

Correlation Functions of Harish-Chandra Integrals over the Orthogonal and the Symplectic Groups

A. Prats Ferrer · B. Eynard · P. Di Francesco · J.-B. Zuber

Received: 15 November 2006 / Accepted: 23 May 2007 / Published online: 7 July 2007

Abstract The Harish-Chandra correlation functions, i.e. integrals over compact groups of invariant monomials $\prod \mathrm{tr}(X^{p_1}\Omega Y^{q_1}\Omega^\dagger X^{p_2}\cdots)$ with the weight $\exp \mathrm{tr}(X\Omega Y\Omega^\dagger)$ are computed for the orthogonal and symplectic groups. We proceed in two steps. First, the integral over the compact group is recast into a Gaussian integral over strictly upper triangular complex matrices (with some additional symmetries), supplemented by a summation over the Weyl group. This result follows from the study of loop equations in an associated two-matrix integral and may be viewed as the adequate version of Duistermaat–Heckman's theorem for our correlation function integrals. Secondly, the Gaussian integration over triangular matrices is carried out and leads to compact determinantal expressions.

1 Introduction

In the study of matrix integrals [1–4], one frequently encounters integrals of the form

$$Z^G = \int d\Omega \mathrm{e}^{-\mathrm{tr}(X\Omega Y\Omega^{-1})} \tag{1.1}$$

A. Prats Ferrer (✉) · B. Eynard · P. Di Francesco · J.-B. Zuber
Service de Physique Théorique de Saclay, CEA/DSM/SPhT, CNRS/SPM/URA 2306,
91191 Gif-sur-Yvette Cedex, France
e-mail: prats@lpthe.jussieu.fr

B. Eynard
e-mail: bertrand.eynard@cea.fr

P. Di Francesco
e-mail: philippe@spht.saclay.cea.fr

J.-B. Zuber
e-mail: zuber@lpthe.jussieu.fr

A. Prats Ferrer · J.-B. Zuber
Université Pierre et Marie Curie–Paris6, CNRS UMR 7589, LPTHE Tour 24-25 5ème étage,
4 Place Jussieu, 75252 Paris Cedex 5, France

over some compact matrix group G, with X and Y two given matrices. By the left and right invariance of the Haar measure $d\Omega$, this integral is invariant under

$$X \to \Omega_1 X \Omega_1^{-1}, \qquad Y \to \Omega_2 Y \Omega_2^{-1}, \tag{1.2}$$

and is thus insensitive to the choice of the representative of the orbits of X and of Y under the (adjoint) action of the group. This enables one to bring the matrices X and Y to some canonical form, as we shall see below.

The case of reference is the so-called Harish-Chandra–Itzykson–Zuber (HCIZ) integral [5, 6], where the integration is performed over the unitary group $\Omega \in \mathrm{U}(n)$ and X and Y are two (anti)Hermitian matrices. By the previous argument, we may with no loss of generality assume that X and Y are two diagonal (anti)Hermitian matrices of size n, $X = \mathrm{diag}(X_i)_{i=1,\dots,n}$, and likewise for Y.

$$\int_{U(n)} d\Omega \mathrm{e}^{-\mathrm{tr}(X\Omega Y\Omega^\dagger)} = \mathrm{const} \cdot \frac{(\det e^{-X_i Y_j})_{1\le i,j\le n}}{\Delta(X)\Delta(Y)} = \mathrm{const} \cdot \sum_{\pi\in\mathfrak{S}_n} \epsilon_\pi \frac{\mathrm{e}^{-\mathrm{tr}(XY^\pi)}}{\Delta(X)\Delta(Y)}, \tag{1.3}$$

where

$$\Delta(X) = \prod_{i<j} (X_i - X_j) \tag{1.4}$$

is the Vandermonde determinant of the eigenvalues X_i of X and likewise for $\Delta(Y)$; $Y^\pi = \mathrm{diag}(Y_{\pi(i)})_{i=1,\dots,n}$. Further examples are provided by more general Harish-Chandra-type integrals, where X and Y live in (a matrix representation of) the Lie algebra $\mathfrak{g}$ of G [5]. For example, $G = O(n)$, X and Y *anti*symmetric real matrices of size n. In all these cases, explicit formulae are known, following from a diversity of methods, see below and for example [7] for a review and references.

It is desirable to extend these formulae to the "correlation functions" of the integral (1.1), i.e. integrals of the form

$$\int d\Omega F(X, \Omega Y \Omega^{-1}) \mathrm{e}^{-\mathrm{tr}(X\Omega Y\Omega^{-1})} \tag{1.5}$$

with F invariant under (1.2). Such correlation functions provide a deeper probe of these integrals and, in a physical context, give often access to quantities of interest. They also act as generating functions of integrals of the form

$$\int d\Omega \, \Omega_{i_1 j_1} \Omega_{i_2 j_2} \cdots \Omega_{i_p j_p} \Omega^{-1}_{k_1 l_1} \cdots \Omega^{-1}_{k_p l_p} \mathrm{e}^{-\mathrm{tr}(X\Omega Y\Omega^{-1})} \tag{1.6}$$

i.e. of *moments* of Ω and $\Omega^\dagger$ with the Harish-Chandra weight.

Kogan et al. [8], Morozov [9], and Shatashvili [10] made some attempts at computing correlation functions of the HCIZ integral, i.e. for the unitary group. Morozov's formula may be recast into a very compact expression [11] but it is only good at computing correlators quadratic in Ω, whilst Shatashvili's formula allows in principle to compute all correlators, but is not of easy use. In paper [12] two of us have shown how to recast the computation of correlators of the type (1.5) for the unitary group into a totally different setting. The method consists in two steps. In step one, the integral is rewritten as a sum of integrals over *upper triangular complex* matrices. The formulae of [12] are in some sense a generalization of Morozov's, and allow to compute all correlators for the $U(n)$ group in a very simple formula. The initial observation was that a Gaussian integral and its polynomial moments in an hyperplane of dimension d of $M_n(\mathbb{C})^2$ does not depend on the hyperplane (up to multiplication by a constant Jacobian), and thus, one can use either the hyperplane $M_1 = M_1^\dagger, M_2 = M_2^\dagger$ or the hyperplane $M_1 = M_2^\dagger$, which have the same dimension. In the first hyperplane, after diagonalization of M_1 and M_2, the integral separates into a radial part and an angular part proportional to moments of $\mathrm{U}(n)$ with the HCIZ measure, while in the second hyperplane, after Schur decomposition of M_1, the integral separates into a radial part (identical to the one in the first hyperplane), a trivial angular part, and a Gaussian integral over triangular matrices. As a result, the authors of [12] were able to identify all moments of the HCIZ integral with a Gaussian integral over complex strictly upper triangular matrices. The formula reads:

$$\begin{aligned}\int_{U(n)} dU\, F(X, UYU^\dagger) \mathrm{e}^{-\operatorname{tr}(XUYU^\dagger)} \\ = \frac{c_n}{\Delta(X)\Delta(Y)} \sum_{\sigma\in\Sigma_n} (-1)^\sigma \mathrm{e}^{-\operatorname{tr}(XY_\sigma)} \int_{T_n} dT\, F(X+T, Y_\sigma + T^\dagger)\mathrm{e}^{-\operatorname{tr}(TT^\dagger)},\end{aligned} \tag{1.7}$$

for X and Y two real diagonal matrices.

In a second step, the Gaussian triangular integrals in the right hand side were computed in [12], using Wick's theorem. The computation can be performed explicitly due to the nilpotent properties of T, which ensures that most Wick's pairings actually vanish. The computation is most easily done by recursion on the size n of the matrix, i.e. by integrating out the last column of T. An appropriate basis of all possible polynomial moments F was introduced in [12], and in that basis, it was found that:

$$\int_{T_n} dT\, F(X+T, Y_\sigma + T^\dagger)\mathrm{e}^{-\operatorname{tr}(TT^\dagger)} = \prod_{i=1}^{n} \mathcal{M}(X_i, Y_{\sigma(i)}), \tag{1.8}$$

where F and $\mathcal{M}(x, y)$ are matrices of some size $R!$. The universal matrices $\mathcal{M}(x, y)$ have many remarkable properties, in particular they commute with one another

$$[\mathcal{M}(x, y), \mathcal{M}(x', y')] = 0. \tag{1.9}$$

The purpose of the present article is to generalize the computations of [12] to other classical Lie groups. Specifically, we address the computation of (1.5) for G the orthogonal group $\mathrm{O}(n)$ or the symplectic group $\mathrm{Sp}(2n)$, with X and Y in the Lie algebra of those groups. First, we relate H-C correlators over those groups to Gaussian integrals over some set of triangular matrices, then we compute the latter Gaussian triangular integrals using an appropriate basis, and finally we find that the result can again be written as products of the same matrices $\mathcal{M}(x, y)$ which appeared for $\mathrm{U}(n)$. Our main results are stated in Theorems 4.3, 5.3 and 6.1 below and in Sect. 7, (see (7.2), (7.3) and (7.4)). The results for the unitary, orthogonal and symplectic groups may be expressed in a unified way, in terms of the Weyl group $\mathcal{W}$, Borel subalgebra $\mathfrak{b}$ and positive roots α, in the following

Theorem 1.1

$$\int_G \mathrm{d}\Omega\, F(X^a, \Omega Y^a \Omega^{-1}) \mathrm{e}^{-\operatorname{tr}(X^a \Omega Y^a \Omega^{-1})}$$

$$= c \sum_{w\in\mathcal{W}} \frac{e^{+\operatorname{tr}(Xw(Y))}}{\prod_{\alpha>0}\alpha(X)\alpha(w(Y))} \times \int_{\mathfrak{n}_+=[\mathfrak{b},\mathfrak{b}]} dT\, F(iX+T, iw(Y)+T^\dagger) e^{-\operatorname{tr}(TT^\dagger)} \tag{1.10}$$

for any polynomial function F and for some F-independent constant c; here X^a and Y^a are taken in a Cartan subalgebra, and should thus be thought of as anti-Hermitian matrices with extra symmetries depending on G, (see Sect. 2 for more details), while iX and iY are the purely imaginary diagonal matrices with the same eigenvalues as X^a and Y^a. In that form, the derived Borel subalgebra $\mathfrak{n}_+$ is made of complex strictly upper triangular matrices, also subject to symmetries. It is thus natural to expect these results to extend to any simple compact group G, see Conjecture 8.1.

It is our hope that these results should provide a new insight on the common features of all these integrals.

Our paper is organized as follows. In Sect. 2, we review the known results by Harish-Chandra and Duistermaat–Heckman and set up the notations. In Sect. 3, we show how Gaussian integrals over two matrices with reality properties, either antisymmetric real or antiselfdual real quaternionic, may be equated to Gaussian integrals over one complex matrix constrained by some symmetry requirements. This is established by use of loop equations, on which we provide details in Appendix 2. Sections 4 and 5 then show how separation of the angular variables by diagonalization or Schur decomposition leads us to the desired integrals, which are thus related to integrals over complex triangular matrices (with additional symmetry requirements). Section 6, supplemented by Appendices 4 and 5, is devoted to the actual computation of these integrals over triangular matrices, by means of a recursive method using a diagrammatic method. The final expressions are displayed in Sect. 7, while Sect. 8 contains our concluding remarks and suggestions of further directions worth exploring. Two other appendices make our notations explicit on quaternions (Appendix 1) or give additional details on the calculation of some Jacobians (Appendix 3).

2 Overview of Known Results

2.1 The Harish-Chandra Theorem [5]

Following Harish-Chandra, for G a compact connected Lie group, we denote by Ad the adjoint action of G on its Lie algebra $\mathfrak{g}$, by (X, Y) the nondegenerate invariant inner product on $\mathfrak{g}$, which we take to be the trace of the product XY in our matrix representation, by $\mathfrak{h}\subset\mathfrak{g}$ the Cartan subalgebra, and by $\alpha(X)$ the linear action of a root α on $X\in\mathfrak{h}$. If X and $Y\in\mathfrak{h}$

$$\Delta(X)\Delta(Y)\int_G d\Omega \exp -(X, \operatorname{Ad}(\Omega)Y) = \frac{\langle\pi,\pi\rangle}{|\mathcal{W}|}\sum_{w\in\mathcal{W}}\epsilon(w)\exp -(X, w(Y)), \tag{2.1}$$

where w is summed over the Weyl group $\mathcal{W}$, $\epsilon(w) = (-1)^{\lambda(w)}$, $\lambda(w)$ is the number of reflections generating w, and

$$\Delta(X) = \prod_{\alpha>0}\alpha(X), \tag{2.2}$$

a product over the positive roots of $\mathfrak{g}$. $\Delta(X)$ may be called a *generalized Vandermonde* determinant, since in the case of $U(n)$, it reduces to (1.4) (and (2.1) reduces to (1.3)), while the expressions for the orthogonal and symplectic groups will be given below. The constant $\langle\pi,\pi\rangle$ in the right hand side of (2.1) is computed as follows. Write all the positive roots in an orthonormal basis ε_i of root space, $i=1,\dots,\ell$, with ℓ the rank of $\mathfrak{g}$. Regard $\pi=\prod_{\alpha>0}\alpha$ as a polynomial in the positive roots and expand it on symmetrized tensor products of the ε, $\pi=\sum_{m_i\geq 0}p(m_1,m_2,\dots,m_\ell)\varepsilon_1^{m_1}\cdots\varepsilon_\ell^{m_\ell}$. Then $\langle\pi,\pi\rangle=\sum_{m_i\geq 0}(p(m_1,m_2,\dots,m_\ell))^2\prod_{i=1}^{\ell}m_i!$. For $\mathfrak{g}=su(n)$, one finds $\langle\pi,\pi\rangle=\prod_{j=1}^{n}j!$, while the expression for the other classical groups will be given below.

2.2 The Duistermaat–Heckman Theorem [13, 14]

The Duistermaat–Heckman theorem states that if $\mathcal{M}$ is a symplectic manifold, invariant under a U(1) flow generated by a Hamiltonian H, then for the integral $\int_{\mathcal{M}}e^{iHt}$, the stationary phase approximation is exact: the sum of the values of the integrand at its critical points, weighted by the Gaussian ('one-loop') fluctuations around them, gives the exact integral.

For X and Y in the Lie algebra, consider the integral equation (1.1)

$$Z=\int_G d\Omega\,\mathrm{e}^{-\operatorname{tr}(X\Omega Y\Omega^{-1})}.$$

In such integrals, we first pick a convenient representative of the orbits of elements of the Lie algebra under the adjoint action of G. A theorem of Cartan asserts that any element of the Lie algebra is the conjugate (under the adjoint action) of an element of the Cartan algebra [18]. Thanks to this theorem and to the left and right invariance of the Haar measure $d\Omega$, one may always assume that X and Y lie in the Cartan subalgebra $\mathfrak{h}$. This assumption matches that of Harish-Chandra's theorem. Moreover the integration is then reduced to G/T, T a maximal Abelian subgroup (Cartan torus) commuting with Y, or alternatively, the integration is carried out on the orbit of Y under the action of this quotient. This is a symplectic manifold, to which Duistermaat–Heckman's theorem applies [15–17].

We thus first look for the critical points of the 'action' $\operatorname{tr}(X\Omega Y\Omega^{-1})$ when $\Omega\in G/T$. In other words, we look for solutions in Ω of

$$\delta\operatorname{tr}(X\Omega Y\Omega^{-1})=\operatorname{tr}(\delta\Omega\Omega^{-1}[X,\Omega Y\Omega^{-1}])=0,$$

Since $A:=\delta\Omega\Omega^{-1}$ is arbitrary in $\mathfrak{g}\backslash\mathfrak{h}$, this implies that the component of $[X,\Omega Y\Omega^{-1}]$ in $\mathfrak{g}\backslash\mathfrak{h}$ vanishes. On the other hand, the component of $[X,\Omega Y\Omega^{-1}]$ in $\mathfrak{h}$ also vanishes, since if $B:=[X,\Omega Y\Omega^{-1}]$ were in $\mathfrak{h}$, then $\operatorname{tr}(B)^2=\operatorname{tr}(B[X,\Omega Y\Omega^{-1}])=\operatorname{tr}([B,X]\Omega Y\Omega^{-1})=0$ since X and $B\in\mathfrak{h}$ commute. We thus conclude that $B=0$, i.e. that

$$[X,\Omega Y\Omega^{-1}]=0. \tag{2.3}$$

The critical points are thus the points $\Omega_c\in G/T$ such that (2.3) is satisfied, which for generic $X\in\mathfrak{h}$ means $\Omega_c Y\Omega_c^{-1}\in\mathfrak{h}$, i.e. Ω_c takes the element $Y\in\mathfrak{h}$ to an element $\Omega_c Y\Omega_c^{-1}\in\mathfrak{h}$. If we denote by $\mathcal{W}$ the normalizer of the Cartan torus T quotiented by T, the previous discussion has just proved that the critical points Ω_c of the action are in one-to-one correspondence with elements w of the group $\mathcal{W}$. The group $\mathcal{W}$ is known to be the Weyl group of G ([18], Proposition 15.8). In the sequel, we denote $Y^w=\Omega_c Y\Omega_c^{-1}$ for $w\in\mathcal{W}$.

At this stage, Duistermaat–Heckman's theorem thus tells us that

$$\int_G d\Omega \mathrm{e}^{-\operatorname{tr}(X\Omega Y\Omega^{-1})} = \sum_{w\in\mathcal{W}} \int_{\mathfrak{g}\backslash\mathfrak{h}} dA \mathrm{e}^{-[\operatorname{tr}(Xe^A Y^w e^{-A})]_2} \tag{2.4}$$

where $[\cdots]_2$ means that we retain only up to the quadratic terms in the expansion in powers of $A \in \mathfrak{g}\backslash\mathfrak{h}$.

The final step in the application of Duistermaat–Heckman theorem is thus to compute the second order variation of the action at one of these critical points. For $\Omega = e^A$, $A \in \mathfrak{g}\backslash\mathfrak{h}$,

$$-\operatorname{tr}(Xe^A Y^w e^{-A}) = -\operatorname{tr}(XY^w) + \frac{1}{2}\operatorname{tr}([A, X][A, Y^w]) + \mathrm{o}(A^2). \tag{2.5}$$

We then have to carry out the Gaussian integration

$$\int d^d A \mathrm{e}^{\frac{1}{2}\operatorname{tr}([A,X][A,Y^w])}$$

over the d-dimensional vector A. This (real) dimension $d = \dim\mathfrak{g} - \dim\mathfrak{h} = 2r$ is even and equal to the number of roots of G. We now expand A, X and Y^w in the standard basis $A = \sum A_\alpha E_\alpha$, $X = \sum_i X_i H_i$ and $Y^w = \sum_i Y_i^w H_i$ and use the standard commutation relations and traces $\operatorname{tr}(H_i H_j) = \delta_{ij}$, $\operatorname{tr}(E_\alpha E_\beta) = \delta_{\alpha+\beta,0}$ to get

$$\begin{aligned}\operatorname{tr}([A, X][A, Y^w]) &= \sum_{\alpha,\beta,i,j} A_\alpha A_\beta X_i Y_j^w \alpha^{(i)}\beta^{(j)}\operatorname{tr}(E_\alpha E_\beta)\\ &= -\sum_{i,j,\alpha} A_\alpha A_{-\alpha}\sum_i X_i\alpha^{(i)}\sum_j Y_j^w\alpha^{(j)}\end{aligned}$$

i.e.

$$\operatorname{tr}([A, X][A, Y^w]) = -\sum_\alpha A_\alpha A_{-\alpha}\alpha(X)\alpha(Y^w)$$

with a sum over positive and negative roots. This quadratic form has a signature $(+^r, -^r)$, and upon a suitable contour rotation, the integration over A yields

$$\int d^d A \mathrm{e}^{\frac{1}{2}\operatorname{tr}([A,X][A,Y^w])} = \frac{\text{constant}}{\prod_{\alpha>0}\alpha(X)\alpha(Y^w)}.$$

Putting everything together, we see that we have reconstructed the Harish-Chandra formula.

2.3 Explicit Formulae

It is of course a good exercise to repeat these steps and to write explicit expressions for each of the classical groups $\mathrm{U}(n)$, $\mathrm{O}(n)$ and $\mathrm{Sp}(n)$. The result for $\mathrm{U}(n)$ is well known and has been recalled above. We shall content ourselves in giving the final result for the two latter cases. In the orthogonal case $\mathrm{O}(n)$, we have to distinguish the $n = 2m$ and $n = 2m+1$ cases. In the even case, $G = \mathrm{O}(2m)$, we take the X and Y matrices in the block diagonal form

$$X = \operatorname{diag}\left(\begin{pmatrix} 0 & X_j \\ -X_j & 0\end{pmatrix}_{j=1,\dots,m}\right), \qquad Y = \operatorname{diag}\left(\begin{pmatrix} 0 & Y_j \\ -Y_j & 0\end{pmatrix}_{j=1,\dots,m}\right). \tag{2.6}$$

Then the critical points Ω_c are the product of a permutation τ of the m blocks of Y by a diagonal matrix of signs $\mathrm{diag}(t_1,\dots,t_m)$, where $t_j=\pm\mathrm{Id}_2$, or in other words, the set $\mathcal{W}$ is $\mathfrak{S}_m\times\mathbb{Z}_2^m$. Note that $\mathcal{W}$ is larger than the ordinary Weyl group of $D_m=so(2m)$ type, which is $W=\mathfrak{S}_m\times\mathbb{Z}_2^{m-1}$: this is because changing the sign of *one* Y_j, say Y_1, is performed by conjugation by a matrix made of 2×2 blocks, $\Omega=\mathrm{diag}(\sigma_1,\mathrm{Id}_2,\dots,\mathrm{Id}_2)$, which is in $\mathrm{O}(2m)$ but *not* in $\mathrm{SO}(2m)$. As a result, only an even number of signs may be changed in the latter case, whence the factor $\mathbb{Z}_2^{m-1}$ in the Weyl group. For the $\mathrm{O}(2m)$ group that we consider here, we thus have

$$Z^{(\mathrm{O}(2m))}=\mathrm{const}\cdot\sum_{w=(\tau,\{t_i\})}\frac{\mathrm{e}^{2\sum_i X_iY_i^w}}{\Delta(X)\Delta(Y^w)}=\mathrm{const}\cdot\sum_{\tau\in\mathfrak{S}_m}\varepsilon_\tau\frac{\prod_i(\mathrm{e}^{2X_iY_{\tau_i}}+\mathrm{e}^{-2X_iY_{\tau_i}})}{\Delta(X)\Delta(Y)}$$
$$=\mathrm{const}\cdot\frac{\det(2\cosh(2X_iY_j))_{i,j=1,\dots,m}}{\Delta(X)\Delta(Y)},\tag{2.7}$$

where ε_τ is the signature of the permutation τ, and

$$\Delta(X)=\prod_{i<j}(X_i^2-X_j^2).\tag{2.8}$$

For $n=2m+1$, the calculation proceeds along the same line. We write

$$X=\mathrm{diag}\left(\begin{pmatrix}0 & X_j\\ -X_j & 0\end{pmatrix}_{j=1,\dots,m},0\right),\qquad Y=\mathrm{diag}\left(\begin{pmatrix}0 & Y_j\\ -Y_j & 0\end{pmatrix}_{j=1,\dots,m},0\right).\tag{2.9}$$

The critical points Ω_c are again the product of a permutation τ of the m blocks of B by a matrix of signs, $t_j=\pm\mathrm{Id}_2$, $j=1,\dots,m$, and

$$Z^{(\mathrm{O}(2m+1))}=\mathrm{const}\cdot\sum_{w=(\tau,\{t_i\})}\frac{\mathrm{e}^{2\sum_i X_iY_i^w}}{\Delta(X)\Delta(Y^w)}$$
$$=\mathrm{const}\cdot\sum_{\tau\in\mathfrak{S}_m}\varepsilon_\tau\frac{\prod_i(\mathrm{e}^{2X_iY_{\tau_i}}-\mathrm{e}^{-2X_iY_{\tau_i}})}{\Delta(X)\Delta(Y)}$$
$$=\mathrm{const}\cdot\frac{\det(2\sinh(2X_iY_j))_{i,j=1,\dots,m}}{\Delta(X)\Delta(Y)}\tag{2.10}$$

with now

$$\Delta(X)=\prod_{i<j}(X_i^2-X_j^2)\prod_{i=1}^m X_i.\tag{2.11}$$

Finally for the symplectic group $\mathrm{Sp}(2m)$, it is convenient to use quaternionic notations for matrices, i.e. to regard the matrix elements as quaternions,[1] with coordinates in the standard quaternion basis, $e_0^2=1$; $e_i^2=-1$, $i=1,\dots,3$, $e_1e_2=e_3$; alternatively, the matrices may be regarded as made of 2×2 blocks written in terms of the identity matrix Id_2 and of Pauli matrices $\vec\sigma$ (with the identification $e_0\leftrightarrow\mathrm{Id}_2$, $e_j\leftrightarrow -i\sigma_j$, $j=1,2,3$). The Lie algebra C_m of $\mathrm{Sp}(2m)$ is thus generated by quaternionic real and anti-Hermitean (also called antiselfdual

[1] We refer the reader to Appendix 1 for more details on our notations on quaternions.

quaternionic real, see Appendix 1) $m\times m$ matrices X, $X_{ij}=X^0_{ij}+\vec{X}_{ij}\,\vec{e}$, $X^\alpha_{ij}\in\mathbb{R}$, $X=-X^\dagger$. Consider the Cartan algebra generated by the m matrices $\mathrm{diag}(X_j e_2)$, $j=1,\dots,m$. We thus take our matrices X and Y of that form

$$X=\mathrm{diag}(X_j e_2)_{j=1,\dots,m},\qquad Y=\mathrm{diag}(Y_j e_2)_{j=1,\dots,m}. \tag{2.12}$$

Then, the critical values Ω_c are again the product of a permutation τ of the m blocks of Y by a diagonal matrix of signs, $t_j=\pm1$. This leads to

$$\begin{aligned} Z^{(\mathrm{Sp}(2m))} &= \mathrm{const}\cdot\sum_{w=(\tau,\{t_i\})}\frac{\mathrm{e}^{2\sum_i X_i Y_i^w}}{\Delta(X)\Delta(Y^w)} \\ &= \mathrm{const}\cdot\sum_{\tau\in\mathfrak{S}_m}\varepsilon_\tau\frac{\prod_i(\mathrm{e}^{2X_iY_{\tau_i}}-\mathrm{e}^{-2X_iY_{\tau_i}})}{\Delta(X)\Delta(Y)} \qquad (2.13)\\ &= \mathrm{const}\cdot\frac{\det(2\sinh(2X_iY_j))_{i,j=1,\dots,m}}{\Delta(X)\Delta(Y)}, \qquad (2.14)\end{aligned}$$

with the same expression for $\Delta(X)$ as in (2.11). Thus $Z^{(\mathrm{Sp}(2m))}$ has the same form as the integral over $\mathrm{O}(2m+1)$.

2.4 List of Notations

For the sake of the reader, we list hereafter the non standard notations in the order they appear in the text.

$\Delta(X)$	Vandermonde determinant and generalizations	(1.4), (2.2), (2.8), (2.11)
$\mathcal{A}_n$	$n\times n$ m real antisymmetric matrices	(3.1)
J	antidiagonal identity matrix	(3.3)
$J\mathcal{A}_n$	$n\times n$ J-antisymmetric complex matrices	Sect. 3.1.2 and (3.4)
$Q\mathcal{A}_m$	$m\times m$ real quaternionic antiselfdual matrices	Sect. 3.2.1 and (3.7)
$\tilde{J}$	antidiagonal symplectic matrix	(3.11)
$\tilde{J}\mathcal{A}_{2m}$	$2m\times 2m$ $\tilde{J}$-antisymmetric complex matrices	Sect. 3.2.2 and (3.13)
$D^a_n(\mathbb{R})$	real 2×2 block-diagonal antisymmetric $n\times n$ matrices	Sect. 4.1
$M_n(\mathbb{C})$	$n\times n$ complex matrices	Sect. 4.2
T_n	$n\times n$ strictly upper triangular complex matrices	Sect. 4.2
$D_n(\mathbb{C})$	$n\times n$ complex diagonal matrices	Sect. 4.2
$\mathrm{U}^J(n)$	twisted orthogonal matrices	(4.5)
T^J_n	$n\times n$ strictly upper triangular J-antisymmetric complex matrices	(4.6)
$D^J_n(\mathbb{C})$, $D^J_n(\mathbb{R})$	$n\times n$ complex, resp. real, J-antisymmetric diagonal matrices	(4.7), (4.18)
$D^{aR}_m(\mathbb{H})$	$m\times m$ real quaternionic diagonal matrices with elements proportional to e_2	Sect. 5.1
$\mathrm{U}^{\tilde{J}}(2m)$	twisted symplectic matrices	(5.6)
$T^{\tilde{J}}_{2m}$	$2m\times 2m$ strictly upper triangular $\tilde{J}$-antisymmetric complex matrices	(5.7)

3 Analytical Continuation for Two-Matrix Integrals

In this section, we follow the same strategy as used in [12] for the unitary group: the integrals of interest (1.5) are regarded as the "angular part" of two-matrix integrals over the classical Lie algebras $so(2m+1)$, $so(2m)$ and $sp(2m)$, and the latter may be analytically continued to integrals over complex matrices with special symmetries.

3.1 Real Antisymmetric Two Matrix Integral and Complex J-Antisymmetric Matrix Integral

3.1.1 Real Antisymmetric Two Matrix Integral

Consider first the set $\mathcal{A}_n$ of $n \times n$ real antisymmetric matrices and consider the measure on $\mathcal{A}_n \times \mathcal{A}_n$

$$\begin{aligned} d\mu(A_1, A_2) &= e^{-\operatorname{tr}(\frac{\alpha_1}{2}A_1^2 + \frac{\alpha_2}{2}A_2^2 + \gamma A_1 A_2)} dA_1 dA_2, \\ dA_k &= \prod_{1 \le i < j \le n} d(A_k)_{i,j}; \quad k = 1, 2. \end{aligned} \tag{3.1}$$

Then the real antisymmetric two matrix partition function and the associated correlation functions are defined as

$$Z_{2RA} = \int_{\mathcal{A}_n \times \mathcal{A}_n} d\mu(A_1, A_2),$$

$$\langle F(A_1, A_2) \rangle_{2RA} = \frac{1}{Z_{2RA}} \int_{\mathcal{A}_n \times \mathcal{A}_n} F(A_1, A_2) d\mu(A_1, A_2)$$

using the measure $d\mu(A_1, A_2)$ given in (3.1). The partition function is the product of $n(n-1)/2$ uncoupled and equal integrals over the pairs of matrix elements $((A_1)_{ij}, (A_2)_{ij})$, $i < j$. Each integral, of the form $\int dx dy \exp\{(x, y) Q (x, y)^T\}$, $Q = \left(\begin{smallmatrix} \alpha_1 & \gamma \\ \gamma & \alpha_2 \end{smallmatrix}\right)$, is absolutely convergent if the real part of the quadratic form $(x, y) Q (x, y)^T := \alpha_1 x^2 + \alpha_2 y^2 + 2\gamma x y$ is negative definite, which holds true if $\operatorname{Re}\alpha_1 \operatorname{Re}\alpha_2 - (\operatorname{Re}\gamma)^2 > 0$ and $\operatorname{Re}\alpha_1, \operatorname{Re}\alpha_2 < 0$ when x and y are integrated over the real line. Then the partition function is easily computed to be

$$Z_{2RA} = \left(\frac{\pi}{\sqrt{\delta}} \right)^{\frac{n(n-1)}{2}} \tag{3.2}$$

where $\delta = \alpha_1 \alpha_2 - \gamma^2$. Likewise, for polynomial $F(A_1, A_2)$, the correlation function $\langle F \rangle_{2RA}$ is by Wick theorem a *polynomial* in the matrix elements of the propagator Q^{-1}, namely $\frac{\alpha_1}{\delta}$, $\frac{\alpha_2}{\delta}$ and $\frac{\gamma}{\delta}$.

3.1.2 Complex J-Antisymmetric Matrix Integral

Define now the $n \times n$ antidiagonal matrix $J = J^{-1}$

$$J = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}. \tag{3.3}$$

Any matrix M with the property $JM^T = -MJ$ is said to be *J-antisymmetric*. Such a matrix is antisymmetric with respect to the second diagonal, i.e. $M_{i,j} = -M_{n+1-j,n+1-i}$

$$\begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,n-1} & 0 \\ M_{2,1} & & \cdots & \iddots & -M_{1,n-1} \\ \vdots & & \iddots & & \vdots \\ M_{n-1,1} & \iddots & & & -M_{1,2} \\ 0 & -M_{n-1,1} & \cdots & -M_{2,1} & -M_{1,1} \end{pmatrix}$$

and in particular, $M_{ij} = 0$ whenever $i + j = n + 1$.
On the set $J\mathcal{A}_n$ of complex J-antisymmetric matrices, we consider the measure

$$\begin{aligned} d\mu(M) &= e^{-\operatorname{tr}(\frac{\alpha_1}{2}M^2 + \frac{\alpha_2}{2}M^{\dagger 2} + \gamma M^\dagger M)} dM, \\ dM &= \prod_{i+j<n+1} d\operatorname{Re} M_{i,j}\, d\operatorname{Im} M_{i,j}. \end{aligned} \tag{3.4}$$

Then the complex J-antisymmetric matrix partition function and the associated correlation functions are defined as

$$Z_{1JA} = \int_{J\mathcal{A}_n} d\mu(M),$$

$$\langle F(M, M^\dagger)\rangle_{1JA} = \frac{1}{Z_{1JA}} \int_{J\mathcal{A}_n} d\mu(M) F(M, M^\dagger)$$

using the measure $d\mu(M)$ given in (3.4). The partition function is again the product of the $n(n-1)/2$ uncoupled and equal integrals over the complex independent matrix elements M_{ij}, $i + j < n + 1$. It is absolutely convergent if $\operatorname{Re}\alpha_1, \operatorname{Re}\alpha_2 > 0$, $\operatorname{Re}\gamma < 0$ and $\operatorname{Re}\gamma^2 > \operatorname{Re}\alpha_1\alpha_2$ and is then given by

$$Z_{1JA} = \left(\frac{\pi}{2\sqrt{-\delta}}\right)^{\frac{n(n-1)}{2}} \tag{3.5}$$

with $\delta = \alpha_1\alpha_2 - \gamma^2$ as before. For polynomial $F(M, M^\dagger)$, the correlation functions $\langle F\rangle_{1JA}$ are again given by polynomials in $\frac{\alpha_1}{\delta}$, $\frac{\alpha_2}{\delta}$ and $\frac{\gamma}{\delta}$.

3.1.3 Analytic Continuation

The two families of integrals just studied have close connections, even though their original domains of convergence may not overlap. The first trivial observation is that in both cases we have the same number of variables, as already manifest in the computations of the partition functions, namely $n(n-1)$ integration variables in both integrals. The second and more important observation for our purpose is that both integrals share the same loop equations. This will be proved in Appendix 2. Third, as already stressed above, the correlations of polynomial invariant functions both in $\mathcal{A}_n \times \mathcal{A}_n$ and in $J\mathcal{A}_n$ are polynomials in the variables $\frac{\alpha_1}{\delta}$, $\frac{\alpha_2}{\delta}$ and $\frac{\gamma}{\delta}$, and thus analytic functions.

From all these observations, we formulate the following

Theorem 3.1 *The polynomial correlation functions of the two real antisymmetric matrix integral are equal to the correlation functions of the complex J-antisymmetric matrix integral in the sense of analytic continuation, i.e.*

$$\langle F(A_1, A_2)\rangle_{2RA} = \langle F(M, M^\dagger)\rangle_{1JA}. \tag{3.6}$$

Proof Note that the loop equations given in Appendix 2 are in fact recursion relations on the polynomial degree of the correlation functions. Then the fact that polynomial invariant correlation functions are polynomials in $\frac{\alpha_1}{\delta}$, $\frac{\alpha_2}{\delta}$ and $\frac{\gamma}{\delta}$ and the fact that the loop equations and their initial condition (namely $\langle 1\rangle = 1$) are the same for both integrals imply that the polynomials generated from the recursion are the same. □

Although the correlation functions are not originally defined in the same region in parameter space, the fact that they are polynomials allows one to analyticly continue them and to identify them.

3.2 Real Quaternionic Antiselfdual Two Matrix Integrals and Complex $\tilde{J}$-Antisymmetric Matrix Integral

In this section we consider another pair of matrix integrals, related to the symplectic group, for which similar considerations hold true.

3.2.1 Real Quaternionic Antiselfdual Two Matrix Integrals

Consider first the set $Q\mathcal{A}_m$ of real quaternionic antiselfdual (anti-Hermitian) $m \times m$ matrices, whose definition has been recalled in Sect. 2.3 and Appendix 1.

On $Q\mathcal{A}_m \times Q\mathcal{A}_m$, we consider the measure given by

$$\begin{aligned} &d\mu(Q_1, Q_2) = e^{-\mathrm{tr}_0(\frac{\alpha_1}{2}Q_1^2+\frac{\alpha_2}{2}Q_2^2+\gamma Q_1Q_2)} dQ_1 dQ_2, \\ &dQ_k = \left(\prod_{i<j}\prod_{\alpha=0}^{3} d(Q_k^{(\alpha)})_{i,j}\right)\left(\prod_{i=1}^{m}\prod_{\alpha=1}^{3} d(Q_k^{(\alpha)})_{i,i}\right); \quad k = 1, 2 \end{aligned} \tag{3.7}$$

where $\mathrm{tr}_0(\dots) = 2\,\mathrm{Re}\,\mathrm{tr}(\dots)$ is a scalar (while $\mathrm{tr}(\dots)$ is in general a quaternion number, see Appendix 1). The quadratic form in this 'Gaussian' measure is thus

$$\begin{aligned} &-\mathrm{tr}_0\left(\frac{\alpha_1}{2}Q_1^2 + \frac{\alpha_2}{2}Q_2^2 + \gamma Q_1 Q_2\right) \\ &\quad = 2\sum_{1\le i<j\le m}\sum_{\alpha=0}^{3}(\alpha_1(Q_1^\alpha)_{ij}^2 + \alpha_2(Q_2^\alpha)_{ij}^2 + 2\gamma(Q_1^\alpha)_{ij}(Q_2^\alpha)_{ij}) \\ &\qquad + \sum_{i=1}^{m}\sum_{\alpha=1}^{3}(\alpha_1(Q_1^\alpha)_{ii}^2 + \alpha_2(Q_2^\alpha)_{ii}^2 + 2\gamma(Q_1^\alpha)_{ii}(Q_2^\alpha)_{ii}). \end{aligned} \tag{3.8}$$

The real quaternionic antiselfdual two matrix partition function and the associated correlation functions are defined as

$$
\begin{aligned}
Z_{2QA} &= \int_{Q\mathcal{A}_m \times Q\mathcal{A}_m} \mathrm{d}\mu(Q_1, Q_2), \\
\langle F(Q_1, Q_2)\rangle_{2QA} &= \frac{1}{Z_{2QA}} \int_{Q\mathcal{A}_n \times Q\mathcal{A}_n} F(Q_1, Q_2)\mathrm{d}\mu(Q_1, Q_2).
\end{aligned}
\tag{3.9}
$$

The partition function is readily computed to be

$$
Z_{2QA} = 2^{3m}\left(\frac{\pi}{2\sqrt{\delta}}\right)^{2m^2+m} \tag{3.10}
$$

and once again correlation functions of polynomials in Q_1, Q_2 are polynomials in $\frac{\alpha_1}{\delta}$, $\frac{\alpha_2}{\delta}$, and $\frac{\gamma}{\delta}$.

3.2.2 Complex $\tilde{J}$-Antisymmetric Matrix Integral

We now introduce a $2m \times 2m$ matrix $\tilde{J} = -\tilde{J}^{-1}$ of the form

$$
\tilde{J} = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \tag{3.11}
$$

written in terms of J defined above in (3.3). Any matrix M with the property $\tilde{J}M^T = -M\tilde{J}$ is said to be *$\tilde{J}$-antisymmetric*. Such a matrix possesses a peculiar symmetry with respect to the second diagonal: we can write it as

$$
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
$$

where A, B, C, D are $m \times m$ matrices satisfying

$$
A = -JD^TJ; \qquad JB^T = BJ; \qquad JC^T = CJ. \tag{3.12}
$$

Thus, under the reflection with respect to the second diagonal, B and C are invariant, while A and $-D$ are exchanged.

On the set $\tilde{J}\mathcal{A}_{2m}$ of complex $\tilde{J}$-antisymmetric matrices we consider the measure

$$
\begin{aligned}
\mathrm{d}\mu(M) &= \mathrm{e}^{-\operatorname{tr}(\frac{\alpha_1}{2}M^2+\frac{\alpha_2}{2}M^{\dagger 2}+\gamma M^\dagger M)}\mathrm{d}M, \\
\mathrm{d}M &= \prod_{i+j\le 2m+1} \mathrm{d}\,\mathrm{Re}\, M_{i,j}\,\mathrm{d}\,\mathrm{Im}\, M_{i,j}.
\end{aligned}
\tag{3.13}
$$

Then the complex $\tilde{J}$-antisymmetric matrix partition function and the associated correlation functions are defined as

$$
\begin{aligned}
Z_{1\tilde{J}A} &= \int_{\tilde{J}\mathcal{A}_{2m}} \mathrm{d}\mu(M), \\
\langle F(M, M^\dagger)\rangle_{1\tilde{J}A} &= \frac{1}{Z_{1\tilde{J}A}} \int_{\tilde{J}\mathcal{A}_{2m}} \mathrm{d}\mu(M)F(M, M^\dagger)
\end{aligned}
\tag{3.14}
$$

using the measure $\mathrm{d}\mu(M)$ given in (3.13). The $\tilde{J}$-antisymmetric partition function reads

$$Z_{1\tilde{J}A} = 2^{2m}\left(\frac{\pi}{2\sqrt{-\delta}}\right)^{2m^2+m}, \tag{3.15}$$

and once again, correlation functions of polynomials in M and $M^\dagger$ are polynomials in the parameters $\frac{\alpha_1}{\delta}$, $\frac{\alpha_2}{\delta}$, and $\frac{\gamma}{\delta}$.

3.2.3 Analytic Continuation

Again the observations made in Sect. 3.1.3 extend to this case. The two matrix integrals of Sects. 3.2.1, 3.2.2 have the same number of integration variables equal to $2m(2m+1)$, they satisfy the same loop equations (see Appendix 2), and their correlation functions of invariant polynomials have polynomial dependence on $\frac{\alpha_1}{\delta}$, $\frac{\alpha_2}{\delta}$ and $\frac{\gamma}{\delta}$.

These observations allow us to formulate an analogous analytic continuation theorem for these two matrix integrals

Theorem 3.2 *The polynomial correlation functions of the two real quaternionic antiselfdual matrix integral are equal to the correlation functions of the complex $\tilde{J}$-antisymmetric matrix integral in the sense of analytic continuation*, *i.e.*

$$\langle F(A_1, A_2)\rangle_{2QA} = \langle F(M, M^\dagger)\rangle_{1\tilde{J}A}. \tag{3.16}$$

Proof The proof goes exactly as the one in Theorem 3.1. □

4 Correlation Functions over the Orthogonal Group

In this section we exploit the relation found in Theorem 3.1 by performing a separation between "angular" and "radial" variables of matrices in the two sides of equation (3.6).

4.1 Block-Diagonalization of Antisymmetric Matrices

We first consider the case of antisymmetric matrices and of the orthogonal group $\mathrm{O}(n)$ equipped with its Haar measure $\mathrm{d}O$ (normalized to $\int \mathrm{d}O = 1$).

As recalled in Sect. 1, Cartan's theorem asserts that any antisymmetric matrix A may be brought to the block diagonal form (2.6) or (2.9) by an orthogonal transformation of $\mathrm{O}(n)$ (for this standard result, see also [19, 21, 22]). Denote by $D_n^a(\mathbb{R})$ the set of such real block-diagonal antisymmetric $n \times n$ matrices, with the Lebesgue measure:

$$\mathrm{d}X := \prod_i^m \mathrm{d}X_i. \tag{4.1}$$

By an abuse of language, we shall refer to the X_i as the "eigenvalues" of A.

In the new variables $\{O, X\}$, the Lebesgue measure in $\mathcal{A}_n$ reads

$$\mathrm{d}A = \mathrm{Jac}_n^O \Delta^2(X)\mathrm{d}O\mathrm{d}X \tag{4.2}$$

where the Jacobian is (see Appendix 3 for details)

$$\mathrm{Jac}_n^O = \begin{cases} \frac{\pi^{m(m-1)}2^{m(m-1)}}{m!\prod_{j=1}^{m-1}(2j)!} & \text{if } n = 2m, \\ \frac{\pi^{m^2}2^{m^2}}{m!\prod_{j=1}^{m}(2j-1)!} & \text{if } n = 2m+1. \end{cases} \tag{4.3}$$

We recall that $\Delta(X)$ takes two different forms (2.8) and (2.11) depending on the parity of n.

This decomposition is unique up to a permutation of the m "eigenvalues", a change of signs of each eigenvalue independently, and a multiplication of O by a 2×2 block-diagonal matrix whose diagonal blocks belong to O(2). In other words, $A = OXO^T$ establishes a mapping between $\mathcal{A}_n$ and $\mathrm{O}(n) \times D_n^a(\mathbb{R})/(\mathrm{O}(2)^m \times \mathfrak{S}_m \times \mathbb{Z}_2^m)$. This overcounting has already been taken into account in Jac_n^O.

4.2 Schur Decomposition of Complex J-Antisymmetric Matrices

A less standard result, (see [19–22] for instance), is that any complex matrix $M \in M_n(\mathbb{C})$ can be written as:

$$M' = U'(Z' + T')U'^{\dagger} \tag{4.4}$$

where $U' \in \mathrm{U}(n)$ is a unitary matrix, $T' \in T_n$ a strictly upper triangular complex matrix and $Z' \in D_n(\mathbb{C})$ a complex diagonal matrix. We can apply this Schur decomposition to a J-antisymmetric matrix. This will induce further constraints on the unitary and triangular matrices.

Define $\mathrm{U}^J(n)$ to be the subgroup of $\mathrm{U}(n)$ satisfying the condition

$$U^{-1} = U^{\dagger} = JU^TJ \tag{4.5}$$

with the induced normalized Haar measure. We will call these matrices *twisted orthogonal* matrices. Define also T_n^J to be the set of $n\times n$ strictly upper triangular J-antisymmetric complex matrices, with the Lebesgue measure:

$$\mathrm{d}T := \prod_{\substack{i<j \\ i+j<N+1}} \mathrm{d}\,\mathrm{Re}\, T_{ij}\,\mathrm{d}\,\mathrm{Im}\, T_{ij} \tag{4.6}$$

and $D_n^J(\mathbb{C})$ to be the set of $n\times n$ complex J-antisymmetric diagonal matrices

$$\begin{aligned} Z &= \mathrm{diag}(Z_1,\dots,Z_m,-Z_m,\dots,-Z_1) \quad \text{or} \\ &\mathrm{diag}(Z_1,\dots,Z_m,0,-Z_m,\dots,-Z_1), \end{aligned} \tag{4.7}$$

depending on the parity of n, with the Lebesgue measure:

$$\mathrm{d}Z := \prod_i^m \mathrm{d}\,\mathrm{Re}\, Z_i\,\mathrm{d}\,\mathrm{Im}\, Z_i. \tag{4.8}$$

Finally we define for these matrices of $D_n^J(\mathbb{C})$

$$\Delta(Z) = \prod_{i<j}(Z_i^2 - Z_j^2) \times \begin{cases} 1 & \text{if } n = 2m, \\ \prod_i^m Z_i & \text{if } n = 2m+1. \end{cases} \tag{4.9}$$

With these notations one can prove

Proposition 4.1 *Any J-antisymmetric matrix M may be written as*:

$$M = U(Z+T)U^{\dagger} \tag{4.10}$$

where $U \in \mathrm{U}^J(n)$, $T \in T_n^J$ and $Z \in D_n^J(\mathbb{C})$. The Lebesgue measure in $J\mathcal{A}_n$ is then:

$$\mathrm{d}M = \mathrm{Jac}_n^{U^J} |\Delta(Z)|^2 \mathrm{d}U \mathrm{d}T \mathrm{d}Z \tag{4.11}$$

where the Jacobian is

$$\mathrm{Jac}_n^{U^J} = \mathrm{Jac}_n^{O} \times \begin{cases} 2^{m-m^2} & \text{if } n = 2m, \\ 2^{-m^2} & \text{if } n = 2m+1. \end{cases} \tag{4.12}$$

Proof Consider the Schur decomposition (4.4) of the matrix M. Noticing that

$$\det(\lambda - M) = \det(\lambda - JMJ) = \det(\lambda + M) \tag{4.13}$$

we immediately see that the non-vanishing eigenvalues come in pairs $(\lambda, -\lambda)$. By a possible redefinition of U, we may always order the eigenvalues in a J-antisymmetric diagonal form Z as in (4.7). The constraints on U and T follow from the J-antisymmetry of M and Z. The measure can be computed using the same method as in the appendices of [19]. □

This decomposition is unique up to a permutation of the m different eigenvalues, to changes of sign of the m eigenvalues and to multiplication of U by a diagonal matrix $V \in \mathrm{U}^J(n)$ whose elements are on the unit circle. In other words, $M = U(Z+T)U^{\dagger}$ provides a 1-to-1 mapping between $J\mathcal{A}_n(\mathbb{C})$ and $\mathrm{U}^J(n) \times T_n^J \times D_n^J(\mathbb{C})/(\mathrm{U}(1)^m \times \mathfrak{S}_m \times \mathbb{Z}_2^m)$. The overcounting is included in (4.11).

4.3 Orthogonal and Triangular Matrix Integrals

4.3.1 Radial and Angular Integrals

Consider the block-diagonal decomposition of the real antisymmetric matrices and the Schur decomposition of the J-antisymmetric complex matrices. Using these decompositions we will rewrite both sides of (3.6).

Theorem 4.1 *A matrix integral over $\mathcal{A}_n \times \mathcal{A}_n$ can be decomposed into a "radial" and an "angular" part using the block-diagonal decomposition $A = OXO^T$.*

$$\begin{aligned} &\int_{\mathcal{A}_n \times \mathcal{A}_n} \mathrm{d}\mu(A_1, A_2) F(A_1, A_2) \\ &\quad = (\mathrm{Jac}_n^O)^2 \int_{D_n^a(\mathbb{R}) \times D_n^a(\mathbb{R})} \mathrm{d}X \mathrm{d}Y \Delta^2(X) \Delta^2(Y) \\ &\qquad \times \mathrm{e}^{-\mathrm{tr}(\frac{\alpha_1}{2}X^2 + \frac{\alpha_2}{2}Y^2)} \int_{\mathrm{O}(n)} \mathrm{d}O\, F(X, OYO^T) \mathrm{e}^{-\gamma\, \mathrm{tr}(XOYO^T)} \end{aligned} \tag{4.14}$$

with the notations of (2.8) *and* (2.11).

Proof The theorem follows from the results of Sect. 4.1. □

Notice that one of the two orthogonal matrices decouples and so this part of the integral gives 1. The remaining orthogonal integral represents the relative angular variables.

Theorem 4.2 *A matrix integral over $J\mathcal{A}_n$ can be decomposed into a "radial", an "angular" and a "triangular" part using the Schur decomposition $M = U(Z+T)U^\dagger$.*

$$\begin{aligned}&\int_{J\mathcal{A}_n} \mathrm{d}\mu(M) F(M, M^\dagger)\\ &\quad = \mathrm{Jac}_n^{\mathrm{U}^J} \int_{D_n^J(\mathbb{C})} \mathrm{d}Z |\Delta(Z)|^2 \mathrm{e}^{-\operatorname{tr}(\frac{\alpha_1}{2} Z^2 + \frac{\alpha_2}{2} Z^{*2})}\\ &\qquad \times \mathrm{e}^{-\gamma \operatorname{tr}(Z^* Z)} \int_{T_n^J} \mathrm{d}T\, F(Z+T, Z^* + T^\dagger) \mathrm{e}^{-\gamma \operatorname{tr}(T^\dagger T)}\end{aligned} \tag{4.15}$$

with the notations of (4.9).

Proof The theorem follows from the results of Sect. 4.2. □

Notice that the twisted orthogonal matrix decouples and so this part of the integral gives 1. Only the triangular and radial parts remain. Notice also that the measure for the triangular part factors out from that of the radial part and only a Gaussian measure remains for the triangular part.

4.3.2 Relating Integrals over Orthogonal and Triangular Matrices

In this subsection, we relate the HC integral over the orthogonal group $\mathrm{O}(n)$

$$\mathcal{I}_F^{\mathrm{O}(n)} := \int_{\mathrm{O}(n)} \mathrm{d}O \mathrm{e}^{-\gamma \operatorname{tr}(X^a O Y^a O^T)} F(X^a, O Y^a O^T), \tag{4.16}$$

with $X^a, Y^a \in D_n^a(\mathbb{R})$, to an integral over complex upper triangular matrices of $T_n^J(\mathbb{C})$. Note first that $\mathcal{I}_F^{\mathrm{O}(n)}$ is a completely symmetric and even function of the "eigenvalues" X_i and of the Y_i, $i = 1, \dots, m = \lfloor n/2 \rfloor$. This is because any permutation or sign changing matrix acting on either X^a or Y^a may be absorbed into a redefinition of the orthogonal matrix O. In contrast, the integral over complex triangular matrices will have to be symmetrized *by hand*.

To obtain the desired relation between HC-type integrals over the orthogonal group and integrals over the triangular matrices of T_n^J, we shall follow the same steps as in [12], in particular of Lemma A.1 there, which asserts that for any polynomial ω in two variables, one has the relation

$$\frac{\int_{\mathbb{C}} \mathrm{d}z \omega(z, z^*) \mathrm{e}^{-\operatorname{tr}(\frac{\alpha_1}{2} z^2 + \frac{\alpha_2}{2} z^{*2} + \gamma z^* z)}}{\int_{\mathbb{C}} \mathrm{d}z \mathrm{e}^{-\operatorname{tr}(\frac{\alpha_1}{2} z^2 + \frac{\alpha_2}{2} z^{*2} + \gamma z^* Z)}} = \frac{\int_{\mathbb{R}\times\mathbb{R}} \mathrm{d}x \mathrm{d}y \omega(x, y) \mathrm{e}^{-\operatorname{tr}(\frac{\alpha_1}{2} x^2 + \frac{\alpha_2}{2} y^2 + \gamma x y)}}{\int_{\mathbb{R}\times\mathbb{R}} \mathrm{d}x \mathrm{d}y \mathrm{e}^{-\operatorname{tr}(\frac{\alpha_1}{2} x^2 + \frac{\alpha_2}{2} y^2 + \gamma x y)}} \tag{4.17}$$

where we have one complex variable integration on the left hand side and two real variables on the right hand side. This relation may be promoted into the following equality between integrals over *diagonal* matrices

$$\frac{\int_{D_n^J(\mathbb{C})} \mathrm{d}Z \omega(Z, Z^*) \mathrm{e}^{-\operatorname{tr}(\frac{\alpha_1}{2} Z^2 + \frac{\alpha_2}{2} Z^{*2} + \gamma Z^* Z)}}{\int_{D_n^J(\mathbb{C})} \mathrm{d}Z \mathrm{e}^{-\operatorname{tr}(\frac{\alpha_1}{2} Z^2 + \frac{\alpha_2}{2} Z^{*2} + \gamma Z^* Z)}}$$

$$= \frac{\int_{D_n^J(\mathbb{R})\times D_n^J(\mathbb{R})} \mathrm{d}X\mathrm{d}Y\omega(X,Y)\mathrm{e}^{-\operatorname{tr}(\frac{\alpha_1}{2}X^2+\frac{\alpha_2}{2}Y^2+\gamma XY)}}{\int_{D_n^J(\mathbb{R})\times D_n^J(\mathbb{R})} \mathrm{d}X\mathrm{d}Y\mathrm{e}^{-\operatorname{tr}(\frac{\alpha_1}{2}X^2+\frac{\alpha_2}{2}Y^2+\gamma XY)}}. \tag{4.18}$$

We now apply Theorems 4.1 and 4.2 to the two sides of (3.6)

$$\langle F\rangle_{2RA} = \frac{1}{Z_{2RA}}\int_{\mathcal{A}_n\times\mathcal{A}_n} \mathrm{d}A_1\mathrm{d}A_2\mathrm{e}^{-\operatorname{tr}(\frac{\alpha_1}{2}A_1^2+\frac{\alpha_2}{2}A_2^2+\gamma A_1A_2)}F(A_1,A_2) \tag{4.19}$$

$$= \frac{(\mathrm{Jac}_n^O)^2}{Z_{2RA}}\int_{D_n^a(\mathbb{R})\times D_n^a(\mathbb{R})} \mathrm{d}X^a\mathrm{d}Y^a\Delta^2(X^a)\Delta^2(Y^a)\mathrm{e}^{-\operatorname{tr}(\frac{\alpha_1}{2}X^{a2}+\frac{\alpha_2}{2}Y^{a2})}$$

$$\times\int_{\mathrm{O}(n)}\mathrm{d}O\mathrm{e}^{-\gamma\operatorname{tr}(X^aOY^aO^T)}F(X^a,OY^aO^T) \tag{4.20}$$

$$= \langle F\rangle_{1JA} = \frac{1}{Z_{1JA}}\int_{J\mathcal{A}_n}\mathrm{d}M\mathrm{e}^{-\operatorname{tr}(\frac{\alpha_1}{2}M^2+\frac{\alpha_2}{2}M^{\dagger 2}+\gamma MM^\dagger)}F(M,M^\dagger) \tag{4.21}$$

$$= \frac{\mathrm{Jac}_n^{U^J}Z_{D_n^J(\mathbb{C})}}{Z_{1JA}}\frac{1}{Z_{D_n^J(\mathbb{C})}}\int_{D_n^J(\mathbb{C})}\mathrm{d}Z|\Delta(Z)|^2\mathrm{e}^{+\operatorname{tr}(\frac{\alpha_1}{2}Z^2+\frac{\alpha_2}{2}Z^{*2}-\gamma ZZ^*)}$$

$$\times\int_{T_n^J}\mathrm{d}T\ \mathrm{e}^{-\gamma\operatorname{tr}(TT^\dagger)}F(iZ+T,-iZ^*+T^\dagger). \tag{4.22}$$

In the last line, we have performed a change of variables $Z\to iZ$ for reasons that will appear soon. We then apply (4.18) to get

$$\langle F\rangle_{1JA} = \frac{\mathrm{Jac}_n^{U^J}Z_{D_n^J(\mathbb{C})}}{Z_{1JA}Z_{D_n^J(\mathbb{R})\times D_n^J(\mathbb{R})}}\int_{D_n^J(\mathbb{R})\times D_n^J(\mathbb{R})}\mathrm{d}X\mathrm{d}Y\Delta(X)\Delta(Y)\mathrm{e}^{\operatorname{tr}(\frac{\alpha_1}{2}X^2+\frac{\alpha_2}{2}Y^2-\gamma XY)}$$

$$\times\int_{T_n^J}dT\mathrm{e}^{-\gamma\operatorname{tr}(TT^\dagger)}F(iX+T,-iY+T^\dagger) \tag{4.23}$$

$$= \frac{\mathrm{Jac}_n^{U^J}Z_{D_n^J(\mathbb{C})}}{Z_{1JA}Z_{D_n^J(\mathbb{R})\times D_n^J(\mathbb{R})}2^m m!}\int_{D_n^J(\mathbb{R})\times D_n^J(\mathbb{R})}\mathrm{d}X\mathrm{d}Y\Delta^2(X)\Delta^2(Y)\mathrm{e}^{\operatorname{tr}(\frac{\alpha_1}{2}X^2+\frac{\alpha_2}{2}Y^2)}$$

$$\times\sum_{\substack{\tau\in\mathfrak{S}_m\\ t\in\mathbb{Z}_2^m}}\frac{\mathrm{e}^{-\gamma\operatorname{tr}(XY_{(\tau,t)})}}{\Delta(X)\Delta(Y_{(\tau,t)})}\int_{T_n^J}dT\mathrm{e}^{-\gamma\operatorname{tr}(TT^\dagger)}F(iX+T,-iY_{(\tau,t)}+T^\dagger). \tag{4.24}$$

In the last line, we have symmetrized the integral over triangular matrices for the reason explained at the beginning of this subsection. In these expressions, $Z_{D_n^J(\mathbb{R})\times D_n^J(\mathbb{R})} = (\pi/\sqrt{\delta})^m$ and $Z_{D_n^J(\mathbb{C})} = (\pi/2\sqrt{-\delta})^m$. We finally compare the integrands of the second and the last lines (4.19) and (4.23) of the previous equation that we rewrite as

$$\int_{D_n^a(\mathbb{R})\times D_n^a(\mathbb{R})}\mathrm{d}X^a\mathrm{d}Y^a\Delta^2(X^a)\Delta^2(Y^a)\mathrm{e}^{-\operatorname{tr}(\frac{\alpha_1}{2}X^{a2}+\frac{\alpha_2}{2}Y^{a2})}$$

$$\times\frac{(\mathrm{Jac}_n^O)^2}{Z_{2RA}}\int_{\mathrm{O}(n)}\mathrm{d}O\mathrm{e}^{-\gamma\operatorname{tr}(X^aOY^aO^T)}F(X^a,OY^aO^T)$$

$$= \int_{D_n^J(\mathbb{R})\times D_n^J(\mathbb{R})}\mathrm{d}X\mathrm{d}Y\Delta^2(X)\Delta^2(Y)\mathrm{e}^{\operatorname{tr}(\frac{\alpha_1}{2}X^2+\frac{\alpha_2}{2}Y^2)}$$

$$\times \frac{\mathrm{Jac}_n^{U^J} Z_{D_n^J(\mathbb{C})}}{Z_{1JA} Z_{D_n^J(\mathbb{R})\times D_n^J(\mathbb{R})} 2^m m!} \sum_{\substack{\tau\in\mathfrak{S}_m \\ t\in\mathbb{Z}_2^m}} \frac{\mathrm{e}^{-\gamma\,\mathrm{tr}(XY_{(\tau,t)})}}{\Delta(X)\Delta(Y_{(\tau,t)})}$$

$$\times \int_{T_n^J} dT \mathrm{e}^{-\gamma\,\mathrm{tr}(TT^\dagger)} F(iX+T, -iY_{(\tau,t)}+T^\dagger) \tag{4.25}$$

Note the sign difference in the two quadratic forms: if $X \in D_n^J(\mathbb{R})$ has eigenvalues X_i, and $X^a \in D_n^a(\mathbb{R})$ is of the form (2.6) or (2.9), then $\mathrm{tr}(X^{a2}) = -2\sum_{i=1}^m X_i = -\mathrm{tr}(X)^2$, and likewise for Y, so that the Gaussian measures match. This justifies *a posteriori* our change of $Z \to iZ$.

In order to identify the two integrands (the second and fourth lines of (4.25)), we notice that by definition these integrands belong to $L^2(\mathbb{R}^{2m})$ with respect to the measure given by the first and third lines of (4.25). Now we proceed as in [12]: by multiplying $F(X,Y)$ by arbitrary polynomials of X, resp. Y, we may multiply the integrands on both sides by arbitrary symmetric even polynomials of the X_i or of the Y_j. By projecting onto the orthogonal polynomials basis of $L^2(\mathbb{R}^{2m})$ with respect to the measure, we deduce that the integrands must be equal. This gives the

Theorem 4.3 *For any invariant polynomial function $F(.,.)$ and any $X^a, Y^a \in D_n^a(\mathbb{R})$, and X, Y the corresponding matrices in $D_n^J(\mathbb{R})$, one has*:

$$\int_{\substack{\mathrm{O}(n=2m)\\ \mathrm{O}(n=2m+1)}} \mathrm{d}O \mathrm{e}^{-\gamma\,\mathrm{tr}(X^a O Y^a O^T)} F(X^a, OY^aO^T)$$

$$= c_n \sum_{\tau\in\mathfrak{S}_m} \sum_{t\in\mathbb{Z}_2^m} \varepsilon_\tau \frac{\mathrm{e}^{\gamma\,\mathrm{tr}(XY_{(\tau,t)})}}{\Delta(X)\Delta(Y)} \int_{T_n^J} \mathrm{d}T \mathrm{e}^{-\gamma\,\mathrm{tr}(T^\dagger T)} F(iX+T, iY_{(\tau,t)}+T^\dagger)$$

$$\times \begin{cases} 1 \\ \prod_i t_i \end{cases} \tag{4.26}$$

where

$$c_n = \frac{\mathrm{Jac}_n^{U^J} Z_{2RA} Z_{D_n^J(\mathbb{C})}}{(\mathrm{Jac}_n^O)^2 Z_{1JA} Z_{D_n^J(\mathbb{R})\times D_n^J(\mathbb{R})} 2^m m!} = \frac{2^{\frac{n(n-1)}{2}}}{4^m m! \mathrm{Jac}_n^O}. \tag{4.27}$$

In (4.26) the dependence on τ and the signs t_i has been made more explicit, and all t_i changed into their opposite.

4.3.3 Examples

Take as an example the case $F(A,B) = 1$. Then

$$\int_{\substack{\mathrm{O}(2m)\\ \mathrm{O}(2m+1)}} \mathrm{d}O \mathrm{e}^{-\gamma\,\mathrm{tr}(X^a O Y^a O^T)}$$

$$= c_n \sum_{\tau\in\mathfrak{S}_m} \sum_{t\in\mathbb{Z}_2^m} \varepsilon_\tau \frac{\prod_{i=1}^m \mathrm{e}^{2\gamma X_i Y_{(\tau,t)(i)}}}{\Delta(X)\Delta(Y)} \int_{T_n^J} \mathrm{d}T \mathrm{e}^{-\gamma\,\mathrm{tr}(T^\dagger T)} \times \begin{cases} 1, \\ \prod_i t_i. \end{cases}$$

Merging the constant c_n and the triangular integral which decouples into a constant K_n we get just a summation over permutations and signs.

$$\begin{aligned}
&\int_{\substack{\mathrm{O}(2m)\\ \mathrm{O}(2m+1)}} \mathrm{d}O \mathrm{e}^{-\gamma\,\mathrm{tr}(X^a O Y^a O^T)} \\
&= K_n \sum_{\tau\in\mathfrak{S}_m} \sum_{t\in\mathbb{Z}_2^m} \varepsilon_\tau \frac{\prod_{i=1}^m \mathrm{e}^{2\gamma X_i Y_{(\tau,t)(i)}}}{\Delta(X)\Delta(Y)} \times \begin{cases} 1, \\ \prod_i t_i \end{cases} \\
&= \frac{K_n}{\Delta(X)\Delta(Y)} \sum_{\tau\in\mathfrak{S}_m} \varepsilon_\tau \prod_{i=1}^m [\mathrm{e}^{2\gamma X_i Y_{\tau(i)}} \pm \mathrm{e}^{-2\gamma X_i Y_{\tau(i)}}] \\
&= \frac{K_n}{\Delta(X)\Delta(Y)} \times \begin{cases} \det 2\cosh(2\gamma X_i Y_j), \\ \det 2\sinh(2\gamma X_i Y_j) \end{cases}
\end{aligned} \tag{4.28}$$

with

$$K_n = c_n \int_{T_n^J} \mathrm{e}^{-\gamma\,\mathrm{tr}(T^\dagger T)}. \tag{4.29}$$

Then with

$$\int_{T_n^J} \mathrm{e}^{-\gamma\,\mathrm{tr}(T^\dagger T)} = \begin{cases} \left(\frac{\pi}{2\gamma}\right)^{m(m-1)} & \text{if } n = 2m, \\ \left(\frac{\pi}{2\gamma}\right)^{m^2} & \text{if } n = 2m+1 \end{cases} \tag{4.30}$$

we obtain

$$K_n = \begin{cases} \frac{\prod_{j=1}^{m-1}(2j)!}{2^m \gamma^{m(m-1)}} & \text{if } n = 2m, \\ \frac{\prod_{j=1}^{m}(2j-1)!}{2^m \gamma^{m^2}} & \text{if } n = 2m+1. \end{cases} \tag{4.31}$$

This is exactly what was obtained by the Duistermaat–Heckman theorem in Sect. 1.3 and serves as a check of our formulae.

5 Correlation Functions over the Symplectic Group

In this section we repeat the analysis made in Sect. 4, in the case related to the symplectic group $\mathrm{Sp}(2m)$ of $2m \times 2m$ symplectic matrices and to Theorem 3.2. Following the same steps we perform the separation between "angular" and "radial" variables of matrices in both sides of (3.16).

5.1 Diagonalization of Real Quaternion Antiselfdual Matrices

We consider the set $D_m^{aR}(\mathbb{H})$ of real quaternion diagonal $m \times m$ matrices whose diagonal elements are real quaternions proportional to e_2 (see (2.12)), with the Lebesgue measure:

$$\mathrm{d}X := \prod_i^m \mathrm{d}X_i. \tag{5.1}$$

Appealing again to Cartan's theorem, as we did in Sect. 1.4, any real antiselfdual quaternion matrix $Q \in Q\mathcal{A}_m$ may be written under the form

$$Q = SXS^\dagger \tag{5.2}$$

where $S \in \mathrm{Sp}(2m)$ and $X \in D^{aR}_{2m}(\mathbb{H})$.

The Lebesgue measure in $Q\mathcal{A}_m$ is then:

$$\mathrm{d}Q = \mathrm{Jac}^{Sp}_{2m} \Delta^2(X)\mathrm{d}S\mathrm{d}X \tag{5.3}$$

with (see Appendix 3)

$$\mathrm{Jac}^{Sp}_{2m} = \frac{\pi^{m^2} 2^m}{m! \prod_{j=1}^m (2j-1)!} \tag{5.4}$$

and $\Delta(X)$ as in (2.11).

This decomposition is unique up to a permutation of the m eigenvalues, up to a change of sign of each eigenvalue independently, and up to multiplication of S by a diagonal quaternion matrix $V \in \mathrm{Sp}(2m)$ whose diagonal elements v_i satisfy

$$v_i = \cos\theta_i + \sin\theta_i e_2; \quad \theta_i \in [0, 2\pi). \tag{5.5}$$

The latter matrices generate a group isomorphic to $\mathrm{O}(2)^m$. In other words, $Q = SXS^\dagger$ provides a 1-to-1 mapping between $Q\mathcal{A}_m$ and $\mathrm{Sp}(2m) \times D^{aR}_m(\mathbb{H})/(\mathrm{O}(2)^m \times \mathfrak{S}_m \times \mathbb{Z}_2^m)$.

5.2 Schur Decomposition of Complex $\tilde{J}$-Antisymmetric Matrices

Let $\mathrm{U}^{\tilde{J}}(2m)$ be the subgroup of $\mathrm{U}(2m)$ unitary group satisfying the condition

$$U^{-1} = U^\dagger = \tilde{J} U^T \tilde{J}^{-1} \tag{5.6}$$

with the induced normalized Haar measure. We will call these matrices *twisted symplectic matrices*.

Define also $T^{\tilde{J}}_{2m}$ to be the set of $2m \times 2m$ strictly upper triangular $\tilde{J}$-antisymmetric complex matrices, with the Lebesgue measure:

$$\mathrm{d}T := \prod_{\substack{i<j \\ i+j \le 2m+1}} d\,\mathrm{Re}\, T_{ij} d\, \mathrm{Im}\, T_{ij}. \tag{5.7}$$

The $2m \times 2m$ J-antisymmetric complex diagonal matrices of $D^J_{2m}(\mathbb{C})$, (see Sect. 4.2), are also $\tilde{J}$-antisymmetric and come with the Lebesgue measure (4.8). Then we prove

Proposition 5.1 *Any $\tilde{J}$-antisymmetric complex matrix M may always be written as*

$$M = U(Z+T)U^\dagger \tag{5.8}$$

where $U \in \mathrm{U}^{\tilde{J}}(2m)$, $T \in T^{\tilde{J}}_{2m}$ and $Z \in D^J_{2m}(\mathbb{C})$.

The Lebesgue measure in $\tilde{J}\mathcal{A}_n$ is then:

$$\mathrm{d}M = \mathrm{Jac}^{\mathrm{U}^{\tilde{J}}}_{2m} |\Delta(Z)|^2 \mathrm{d}U\mathrm{d}T\mathrm{d}Z \tag{5.9}$$

where again $\Delta(Z)=\prod_{i<j}(Z_i^2-Z_j^2)\prod_i^m Z_i$ *and*

$$\mathrm{Jac}_{2m}^{\mathrm{U}^{\tilde J}}=2^{-2m}\mathrm{Jac}_{2m}^{Sp}. \tag{5.10}$$

Proof Similarly to the J-antisymmetric case one can see that

$$\mathrm{Det}(\lambda-M)=\mathrm{Det}(-\lambda-\tilde J M\tilde J)=\mathrm{Det}(\lambda+M) \tag{5.11}$$

so that the eigenvalues come in pairs $(\lambda,-\lambda)$. One may reorder them to make Z as well as M $\tilde J$-antisymmetric and then the constraints on U and T follow. Again the computation of the measure follows the lines of [19]. □

This decomposition is unique up to a permutation of the m different eigenvalues, up to changes of sign of the m eigenvalues and up to multiplication of U by a diagonal matrix $V\in\mathrm{U}^{\tilde J}(2m)$ whose elements are on the unit circle.

In other words, $M=U(Z+T)U^\dagger$ provides a 1-to-1 mapping between $\tilde J\mathcal{A}_{2m}$ and $\mathrm{U}^{\tilde J}(2m)\times T_{2m}^{\tilde J}\times D_{2m}^{\tilde J}(\mathbb{C})/(\mathrm{U}(1)^m\times\mathfrak{S}_m\times\mathbb{Z}_2^m)$.

5.3 Symplectic and Triangular Matrix Integrals

5.3.1 Radial and Angular Integrals

Consider the diagonal decomposition of the real antiselfdual quaternion matrices and the Schur decomposition of the $\tilde J$-antisymmetric complex matrices. Using them we will rewrite both sides of (3.16). With the measure $\mathrm{d}\mu(Q_1,Q_2)$ defined in (3.7), we have the

Theorem 5.1 *A matrix integral over* $Q\mathcal{A}_m\times Q\mathcal{A}_m$ *can be decomposed into a "radial" and an "angular" part using the diagonal decomposition* $Q=SXS^\dagger$. *We have*:

$$\begin{aligned}&\int_{Q\mathcal{A}_m\times Q\mathcal{A}_m}\mathrm{d}\mu(Q_1,Q_2)F(Q_1,Q_2)\\&\quad=(\mathrm{Jac}_{2m}^{Sp})^2\int_{D_m^{aR}(\mathbb{H})\times D_m^{aR}(\mathbb{H})}\mathrm{d}X\mathrm{d}Y\,\Delta^2(X)\Delta^2(Y)\\&\qquad\times\mathrm{e}^{-\mathrm{tr}_0(\frac{\alpha_1}{2}X^2+\frac{\alpha_2}{2}Y^2)}\int_{\mathrm{Sp}(2m)}\mathrm{d}S\,F(X,SYS^\dagger)\mathrm{e}^{-\gamma\,\mathrm{tr}_0(XSYS^\dagger)}.\end{aligned}$$

Proof The theorem follows from what is explained in Sect. 5.1. □

Notice that one of the two symplectic matrices decouples and so this part of the integral gives 1. The remaining symplectic integral runs over the relative angular variables.

Theorem 5.2 *A matrix integral over* $\tilde J\mathcal{A}_{2m}$ *can be decomposed into a "radial", an "angular" and a "triangular" part using the Schur decomposition* $M=U(Z+T)U^\dagger$. *We have*:

$$\begin{aligned}&\int_{M_{2m}^{\tilde J}(\mathbb{C})}\mathrm{d}\mu(M)F(M,M^\dagger)\\&\quad=\mathrm{Jac}^{\mathrm{U}_{2m}^{\tilde J}}\int_{D_{2m}^{\tilde J}(\mathbb{C})}\mathrm{d}Z|\Delta(Z)|^2\mathrm{e}^{-\mathrm{tr}(\frac{\alpha_1}{2}Z^2+\frac{\alpha_2}{2}Z^{*2})}\mathrm{e}^{-\gamma\,\mathrm{tr}(Z^*Z)}\\&\qquad\times\int_{T_{2m}^{\tilde J}}\mathrm{d}T\,F(Z+T,Z^*+T^\dagger)\mathrm{e}^{-\gamma\,\mathrm{tr}(T^\dagger T)}.\end{aligned}$$

Proof The theorem follows from what is explained in Sect. 5.2. □

Notice that the twisted symplectic matrix decouples and so this part of the integral gives 1. Only the triangular and radial parts remain. Notice also that the measure for the triangular and the radial part factors out and so only a Gaussian measure remains for the triangular part.

5.3.2 Relating Integrals over Symplectic and Triangular Matrices

Just as in the orthogonal case, we first observe that the integral over the symplectic group

$$\mathcal{I}^{\mathrm{Sp}(2m)} := \int_{\mathrm{Sp}(2m)} \mathrm{d}S \mathrm{e}^{-\gamma \operatorname{tr}(X) SYS^\dagger} F(X, SYS^\dagger) \tag{5.12}$$

with $X, Y \in D_m^{aR}(\mathbb{H})$, is a completely symmetric and even function of the variables X_i and of the Y_i, $i = 1, \dots, m$, since permutation and sign changing matrices may be absorbed into the symplectic matrix S. Then the same considerations as in Sect. 4.3.2 apply when we want to use (4.18). The Gaussian measure still gets the wrong sign, and the same change of variables $Z \to iZ$ must be used. Then

Theorem 5.3 *For any polynomial invariant function $F(.,.)$, for any $X^a, Y^a \in D_m^{aR}(\mathbb{H})$ and X, Y the associated matrices in $D_{2m}^J(\mathbb{R})$, one has*:

$$\begin{aligned} &\int_{\mathrm{Sp}(2m)} \mathrm{d}S F(X^a, SY^aS^\dagger) \mathrm{e}^{-\gamma \operatorname{tr}_0(X^a SY^a S^\dagger)} \\ &\quad = \tilde{c}_{2m} \sum_{\tau \in \mathfrak{S}_m} \sum_{t \in \mathbb{Z}_2^m} \varepsilon_\tau \prod_{j=1}^m t_j \frac{\mathrm{e}^{\gamma \operatorname{tr}(XY_{(\tau,t)})}}{\Delta(X)\Delta(Y)} \\ &\qquad \times \int_{T_{2m}^{\tilde{J}}} \mathrm{d}T F(iX + T, iY_{(\tau,t)} + T^\dagger) \mathrm{e}^{-\gamma \operatorname{tr}(T^\dagger T)} \end{aligned}$$

with

$$\tilde{c}_{2m} = \frac{(-1)^m \mathrm{Jac}^{\mathrm{U}_{2m}^{\tilde{J}}} Z_{2QA} Z_{D_{2m}^J(\mathbb{C})}}{(\mathrm{Jac}_{2m}^{Sp})^2 Z_{1\tilde{J}A} Z_{D_m^{aR}(\mathbb{H}) \times D_m^{aR}(\mathbb{H})} 2^m m!} = \frac{1}{2^m m! \mathrm{Jac}_{2m}^{O}} \frac{1}{4^m}. \tag{5.13}$$

Proof The proof starts from (3.16), makes use of Theorems 5.1 and 5.2, and then follows the same steps as the proof of Theorem 4.3, including a change of variables $Z \to iZ$, a symmetrization in the variables X_i and Y_j and the use of orthogonal polynomials. □

5.3.3 Examples

Let's take as an example the case $F(A, B) = 1$. Then

$$\begin{aligned} &\int_{\mathrm{Sp}(2m)} \mathrm{d}S \mathrm{e}^{-\gamma \operatorname{tr}(SX^aS^\dagger Y^a)} \\ &\quad = \tilde{c}_{2m} \sum_{\tau \in \mathfrak{S}_m} \sum_{t \in \mathbb{Z}_2^m} \varepsilon_\tau \prod_{j=1}^m t_j \frac{\prod_{i=1}^m \mathrm{e}^{2\gamma t(i) X_i Y_{\tau(i)}}}{\Delta(X)\Delta(Y)} \times \int_{T_{2m}^{\tilde{J}}} \mathrm{d}T \mathrm{e}^{-\gamma \operatorname{tr}(T^\dagger T)} \end{aligned}$$

$$= \tilde{K}_{2m} \sum_{\tau\in\mathfrak{S}_m} \sum_{t\in\mathbb{Z}_2^m} \varepsilon_\tau \prod_{j=1}^m t_j \frac{\prod_{i=1}^m e^{2\gamma t(i) X_i Y_{\tau(i)}}}{\Delta(X)\Delta(Y)}$$

$$= \tilde{K}_{2m} \frac{\det 2\sinh(2\gamma X_i Y_j)}{\Delta(X)\Delta(Y)}$$

with

$$\tilde{K}_{2m} = \tilde{c}_{2m} \int_{T_{2m}^{\tilde{J}}} e^{-\gamma\,\mathrm{tr}(T^\dagger T)}. \tag{5.14}$$

Then with

$$\int_{T_{2m}^{\tilde{J}}} e^{-\gamma\,\mathrm{tr}(T^\dagger T)} = 2^m \left(\frac{\pi}{2\gamma}\right)^{m^2} \tag{5.15}$$

we get

$$\tilde{K}_{2m} = 2^{-(m^2+2m)} \frac{\prod_{j=1}^m (2j-1)!}{2^m} \tag{5.16}$$

which reproduces again the Duistermaat–Heckman result.

In the examples considered in this section and in Sect. 4.3.3, the triangular integrals are just constants, which is not the case in general. We are going to present an explicit formula to compute them.

6 $J/\tilde{J}$-Antisymmetric Triangular Integrals

In order to compute the correlation functions in the orthogonal and the symplectic group we need to compute explicitly various kinds of triangular integrals. For the orthogonal, resp. symplectic, case we have to compute integrals over J-, resp. $\tilde{J}$-, antisymmetric strictly upper triangular complex matrices. We shall unify both kinds of integrals into one formalism and explicitly perform the integration.

6.1 Preliminaries to the Integration

The type of integrals we are interested in are of the form

$$\int_{\mathcal{J}}^{(n)} F(X+T, Y+T^\dagger) e^{-\gamma\,\mathrm{tr}(T^\dagger T)} \mathrm{d}T \tag{6.1}$$

where n is the matrix size, $\mathcal{J}$ stands for J or for $\tilde{J}$, and $\int_{\mathcal{J}}$ refers to whether we integrate over J or $\tilde{J}$-antisymmetric triangular matrices, and X and Y are J- (or $\tilde{J}$-) antisymmetric *diagonal* real matrices. Since the measure is Gaussian it is more convenient to normalize the integrals

$$\langle F(X+T, Y+T^\dagger)\rangle_{\mathcal{J}} = \frac{\int_{\mathcal{J}}^{(n)} F(X+T, Y+T^\dagger) e^{-\gamma\,\mathrm{tr}(T^\dagger T)} \mathrm{d}T}{\int_{\mathcal{J}}^{(n)} e^{-\gamma\,\mathrm{tr}(T^\dagger T)} \mathrm{d}T}. \tag{6.2}$$

From now on we set $2\gamma = 1$, in order to make the propagators simpler.

The typical functions $F(A, B)$ we want to use are constructed from resolvents $\frac{1}{x-A}$ and twisted resolvents $\mathcal{J}(\frac{1}{x-A})^T \mathcal{J}$. These functions are not allowed in general by the analytical continuation theorems, but this is not a problem if we consider that their x series expansions are generating functions of invariant polynomials. An example of such a function is

$$\begin{aligned}&\mathrm{tr}\left(\frac{1}{x_1-(X+T)}\mathcal{J}\left(\frac{1}{y_1-(Y+T^\dagger)}\right)^T\mathcal{J}\right)\\&\quad\times\mathrm{tr}\left(\frac{1}{x_2-(X+T)}\mathcal{J}\left(\frac{1}{y_2-(Y+T^\dagger)}\right)^T\left(\frac{1}{x_3-(X+T)}\right)^T\mathcal{J}\right.\\&\quad\left.\times\frac{1}{y_3-(Y+T^\dagger)}\right).\end{aligned}$$

The procedure we use to compute this integral consists in integrating over the last column (and by symmetry, over the first row) of the triangular matrices, so as to find a recursion on the size n of the matrices, which takes n to $n-2$.

Define the submatrices $\hat{X}$ and $\hat{Y}$ by

$$X=\mathrm{diag}(\alpha,\hat{X},-\alpha);\qquad Y=\mathrm{diag}(\beta,\hat{Y},-\beta),\tag{6.3}$$

and the $\mathcal{J}$-antisymmetric upper-triangular matrices $\hat{T}$ of size $n-2$

$$T=\begin{pmatrix}0 & T_{12}\quad\dots & T_{1n}\\ \vdots & \begin{pmatrix}0 & & \hat{T}\\ \vdots & \ddots & \\ 0 & \dots & 0\end{pmatrix} & \begin{matrix}\vdots\\ T_{n-1n}\end{matrix}\\ 0 & \dots & 0\end{pmatrix}.\tag{6.4}$$

With these definitions and the relations

$$\frac{1}{x-(X+T)}=\frac{1}{x-X}\sum_{n=0}^{\infty}\left(T\frac{1}{x-X}\right)^n=\sum_{n=0}^{\infty}\left(\frac{1}{x-X}T\right)^n\frac{1}{x-X},\tag{6.5}$$

we can expand the resolvent of size n in terms of the resolvent of size $n-2$ and of the variables to be integrated out

$$\begin{aligned}&\left(\frac{1}{x-(X+T)}\right)_{i,j}\\&\quad=\delta_{i,1}\delta_{j,1}\frac{1}{x-\alpha}+\delta_{i,n}\delta_{j,n}\frac{1}{x+\alpha}+\delta_{i,1}\delta_{j,n}\frac{1}{x-\alpha}T_{1,n}\frac{1}{x+\alpha}\\&\qquad+(1-\delta_{i,1}-\delta_{i,n})(1-\delta_{j,1}-\delta_{j,n})\left(\frac{1}{x-(\hat{X}+\hat{T})}\right)_{i,j}\\&\qquad+\delta_{i,1}(1-\delta_{j,1}-\delta_{j,n})\frac{1}{x-\alpha}\left[\sum_{k=2}^{j}T_{1,k}\left(\frac{1}{x-(\hat{X}+\hat{T})}\right)_{k,j}\right]\\&\qquad+(1-\delta_{i,1}-\delta_{i,n})\delta_{j,n}\left[\sum_{l=i}^{n-1}\left(\frac{1}{x-(\hat{X}+\hat{T})}\right)_{i,l}T_{l,n}\right]\frac{1}{x+\alpha}\\&\qquad+\delta_{i,1}\delta_{j,n}\frac{1}{x-\alpha}\left[\sum_{2\le k<l\le n-1}T_{1,k}\left(\frac{1}{x-(\hat{X}+\hat{T})}\right)_{k,l}T_{l,n}\right]\frac{1}{x+\alpha}.\end{aligned}\tag{6.6}$$

Notice that $T_{1,n}$ in the J-antisymmetric case is identically zero, a fact that will be accounted for in the following. In both cases, the only independent integration variables are the matrix elements of the first row. Their propagators are read off the Gaussian weight, which is, in the J-antisymmetric case

$$\mathrm{e}^{-\frac{1}{2}\sum_{i=2}^{n-1}(|T_{1,i}|^2+|T_{i,n}|^2)} = \mathrm{e}^{-\sum_{i=2}^{n-1}|T_{1,i}|^2} \tag{6.7}$$

while for the $\tilde{J}$-antisymmetric case it is

$$\mathrm{e}^{-\frac{1}{2}\sum_{i=2}^{n-1}(|T_{1,i}|^2+|T_{i,n}|^2)-\frac{1}{2}|T_{1,n}|^2} = \mathrm{e}^{-\sum_{i=2}^{n-1}|T_{1,i}|^2-\frac{1}{2}|T_{1,n}|^2}. \tag{6.8}$$

The independent nonzero propagators are thus

$$\begin{aligned} \langle T_{1,i}T^\dagger_{j,1}\rangle &= \delta_{i,j}, \quad 2\le i,j\le n-1, \\ \langle T_{1,n}T^\dagger_{n,1}\rangle &= 1+b \end{aligned} \tag{6.9}$$

where $b=-1$ (resp. $b=+1$) for the J-antisymmetric (resp. $\tilde{J}$-antisymmetric) case, so that in the $b=-1$ case, the propagator for $T_{1,n}$ is zero, as it should. The other propagators encountered in the integration result from the symmetry properties

$$\langle T_{i,n}T^\dagger_{n,j}\rangle = \delta_{i,j}, \qquad \langle T_{1,i}T^\dagger_{n,j}\rangle = -\mathcal{J}_{i,j}, \qquad \langle T_{i,n}T^\dagger_{j,1}\rangle = -(\mathcal{J}^{-1})_{i,j}. \tag{6.10}$$

This is what is needed to perform the first step in the recursive computation of the triangular integrals. Let's take the simplest mixed case.

6.2 Example: Morozov-Like Formula

The simplest case involves two resolvents. Define the two functions

$$\begin{aligned} F_+^{(n)}(x,y,A,B) &= \mathrm{tr}\left(\frac{1}{x-(X+T)}\frac{1}{y-(Y+T^\dagger)}\right)+1, \\ F_-^{(n)}(x,y,A,B) &= \mathrm{tr}\left(\frac{1}{x-(X+T)}\mathcal{J}\left(\frac{1}{y-(Y+T^\dagger)}\right)^T\mathcal{J}\right)+b. \end{aligned}$$

The second one, $F_-^{(n)}$, is *twisted* by the action of $\mathcal{J}$. Using that, for A and B two $\mathcal{J}$-antisymmetric matrices (such as X, Y, T or $\hat{T}$),

$$\begin{aligned} J\left(\frac{1}{x-A}\right)^T J &= \frac{1}{x+A}, \\ \tilde{J}\left(\frac{1}{x-A}\right)^T(-\tilde{J}) &= \frac{1}{x+A}, \end{aligned} \tag{6.11}$$

i.e. $\mathcal{J}(\frac{1}{x-A})^T\mathcal{J} = \frac{b}{-x-A}$, one sees that $F_-^{(n)}(x,y,A,B) = bF_+^{(n)}(x,-y,A,B)$, thus it suffices to carry out the integration over the last column and first row of F_+ only.

The computation goes as follows

$$\left\langle \mathrm{tr}\left(\frac{1}{x-(X+T)}\frac{1}{y-(Y+T^\dagger)}\right)+1\right\rangle_{(n)}$$

$$
\begin{aligned}
&= \left[1 + \frac{1}{x-\alpha}\frac{1}{y-\beta} + \frac{1}{x+\alpha}\frac{1}{y+\beta}\right]\langle 1\rangle_{(n-2)} \\
&\quad + \frac{1}{x^2-\alpha^2}\frac{1}{y^2-\beta^2}\langle 1\rangle_{(n-2)}\langle T_{1,n}T^\dagger_{n,1}\rangle \\
&\quad + \left\langle \mathrm{tr}\left(\frac{1}{x-(\hat{X}+\hat{T})}\frac{1}{y-(\hat{Y}+\hat{T}^\dagger)}\right)\right\rangle_{(n-2)} \\
&\quad + \frac{1}{x-\alpha}\frac{1}{y-\beta}\left\langle\left(\frac{1}{x-(\hat{X}+\hat{T})}\frac{1}{y-(\hat{Y}+\hat{T}^\dagger)}\right)_{k,l}\right\rangle_{(n-2)}\langle T_{1,k}T^\dagger_{l,1}\rangle \\
&\quad + \frac{1}{x+\alpha}\frac{1}{y+\beta}\left\langle\left(\frac{1}{y-(\hat{Y}+\hat{T}^\dagger)}\frac{1}{x-(\hat{X}+\hat{T})}\right)_{k,l}\right\rangle_{(n-2)}\langle T_{l,n}T^\dagger_{n,k}\rangle \\
&\quad + \frac{1}{x^2-\alpha^2}\frac{1}{y^2-\beta^2}\left\langle\left(\frac{1}{x-(\hat{X}+\hat{T})}\right)_{k,l}\left(\frac{1}{y-(\hat{Y}+\hat{T}^\dagger)}\right)_{k',l'}\right\rangle_{(n-2)} \\
&\quad \times \langle T_{1,k}T_{l,n}T^\dagger_{n,k'}T^\dagger_{l',1}\rangle.
\end{aligned}
\tag{6.12}
$$

Inserting the propagators given above we find

$$
\begin{aligned}
&\left\langle \mathrm{tr}\left(\frac{1}{x-(X+T)}\frac{1}{y-(Y+T^\dagger)}\right)+1\right\rangle_{(n)} \\
&\quad = \left[1+\frac{1}{x-\alpha}\frac{1}{y-\beta}\right]\left[1+\frac{1}{x+\alpha}\frac{1}{y+\beta}\right] \\
&\qquad \times \left\langle \mathrm{tr}\left(\frac{1}{x-(\hat{X}+\hat{T})}\frac{1}{y-(\hat{Y}+\hat{T}^\dagger)}\right)+1\right\rangle_{(n-2)} \\
&\qquad + \left[\frac{1}{x^2-\alpha^2}\frac{1}{y^2-\beta^2}\right] \\
&\qquad \times \left\langle \mathrm{tr}\left(\frac{1}{x-(\hat{X}+\hat{T})}\mathcal{J}\left(\frac{1}{y-(\hat{Y}+\hat{T}^\dagger)}\right)^T\mathcal{J}\right)+b\right\rangle_{(n-2)}.
\end{aligned}
\tag{6.13}
$$

We have split the two terms in the $\langle T_{1,n}T^\dagger_{n,1}\rangle = 1+b$ propagator in the following way: the weight 1 goes together with the untwisted minimal cycle and the weight b with a twisted one. This is a general rule as we shall see later.

Notice that we need both functions $F_\pm$ in order to close the recursion relation. Defining the column vector $\mathcal{V}_n = (F_+^{(n)}, F_-^{(n)})^T$, we obtain a recursion formula for the two functions in the form

$$
\mathcal{V}_n = \overline{\mathcal{M}}(x,y,\alpha,\beta)\mathcal{V}_{n-2}
\tag{6.14}
$$

where

$$
\begin{aligned}
&\overline{\mathcal{M}}(x,y,\alpha,\beta) \\
&\quad = \begin{pmatrix} \left(1+\frac{1}{x-\alpha}\frac{1}{y-\beta}\right)\left(1+\frac{1}{x+\alpha}\frac{1}{y+\beta}\right) & \frac{1}{x^2-\alpha^2}\frac{1}{y^2-\beta^2} \\ \frac{1}{x^2-\alpha^2}\frac{1}{y^2-\beta^2} & \left(1+\frac{1}{x-\alpha}\frac{1}{(-y)-\beta}\right)\left(1+\frac{1}{x+\alpha}\frac{1}{(-y)+\beta}\right) \end{pmatrix}.
\end{aligned}
\tag{6.15}
$$

This structure will appear in the general case.

6.3 Last-Row/First-Column Integration: General Case

In the general case, the recursion involve combinations of correlation functions conveniently labeled by graphs.

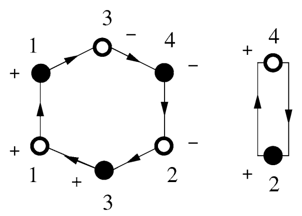
6.3.1 Basis of Correlation Functions

In the example above, we had to mix correlation functions with two resolvents with correlation functions with a lesser number of resolvents, in order to write recursion relations. This will still be necessary in the general case, and we shall see that the basis of correlation functions that we have to consider is conveniently labeled by *tetrads* $\omega = \{\sigma, \tau, s, t\} \in \mathfrak{S}_R \times \mathfrak{S}_R \times \mathbb{Z}_2^R \times \mathbb{Z}_2^R$, made of two permutations of $\mathfrak{S}_R$ of R objects and two sets of R signs. This integer R will turn out to be the maximal number of resolvents of X-type appearing in the correlation function. To each such tetrad, we first associate an oriented bicolored graph G in the following way: G has $2R$ vertices, R of each color; the ith black (resp. white) vertex carry a sign $s(i)$ (resp. $t(i)$), $i = 1, \dots, R$. An oriented edge connects each ith black (resp. white) vertex to the $\sigma(i)$th white (resp. $\tau^{-1}(j)$th black) vertex. The graph is thus made of bicolored cycles.

According to these rules, the graph representation of the tetrad

$$\omega = \{(13)(24), (1)(243), (+, +, +, -), (+, -, -, +)\} \tag{6.16}$$

is



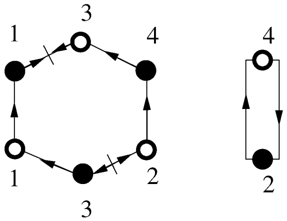
For our purposes we are only interested in the relative signs between vertices in each cycle. We say that two tetrads ω and ω' are equivalent if they are equal up to independent global signs *in each cycle*. We take as representatives of the equivalence classes the tetrads with $s(i) = +$ for the black vertex with the smallest i in each cycle.

There exists also a graphical representation for the equivalence classes. We call diagrams these new objects. Since we only care about the relative signs between vertices in the same cycle, we replace the $\mathbb{Z}_2$ variables by symbols representing changes of sign.[2] We represent this with a short bar across the edges indicating the change of orientation of the edge. In the diagrams, the vertices will be called dots and the edges will be called links. The equivalence class

$$[\omega] = [\{(13)(24), (1)(243), (+, +, +, -), (+, -, -, +)\}] \tag{6.17}$$

is thus represented by the diagram

[2]This operation is accompanied by a change of orientation of edges, and the cycles are no longer oriented. But the original class of oriented graphs may be reconstructed from these diagrams.



In Appendix 5 we prove that the space of equivalence classes of tetrads is isomorphic to the space of permutations of size $2R$, so we can label the equivalence classes $[\omega]$ by an associated permutation $\pi \in \mathfrak{S}_{2R}$. This very important bijection will allow us later to relate the $\mathrm{O}(n)$ and the $\mathrm{Sp}(2m)$ integrals with the U_n group integrals studied in [12].

The correlation functions we need turn out to be labeled by these equivalence classes, or equivalently by permutations of size $2R$. We define in general the $[\omega]$ component of the basis of correlation functions as follows. Consider $[\omega] = [\{\sigma, \tau, s, t\}]$ and consider two sets of variables, $\{x_1, \dots, x_R\}$ for X-type resolvents and $\{y_1, \dots, y_R\}$ for Y-type resolvents. Then we define the following function,

$$F^J_{\{\sigma,\tau,s,t\}}(\{x\},\{y\},A,B)$$
$$:= \prod_{k=1}^{p}\left(t(j_{k,1})\delta_{R_k,1} + \operatorname{tr}\left(\prod_{l=1}^{R_k}\left(\frac{1}{x_{i_{k,l}}-A}\right)^{\zeta^x(i_{k,l})} J^{\eta^{x\to y}_{k,l}} \left(\frac{1}{y_{j_{k,l}}-B}\right)^{\zeta^y(j_{k,l})} J^{\eta^{y\to x}_{k,l}}\right)\right) \tag{6.18}$$

for the J-antisymmetric integrals and

$$F^{\tilde{J}}_{\{\sigma,\tau,s,t\}}(\{x\},\{y\},A,B)$$
$$:= \prod_{k=1}^{p}\left(\delta_{R_k,1} + \operatorname{tr}\left(\prod_{l=1}^{R_k}\left(\frac{1}{x_{i_{k,l}}-A}\right)^{\zeta^x(i_{k,l})} \tilde{J}^{\eta^{x\to y}_{k,l}} \left(\frac{1}{y_{j_{k,l}}-B}\right)^{\zeta^y(j_{k,l})} \tilde{J}^{\eta^{y\to x}_{k,l}}\right)\right) \tag{6.19}$$

for the $\tilde{J}$-antisymmetric ones, where p is the number of cycles of the permutation $\sigma\tau^{-1}$, and R_k is the length of the kth cycle.

The permutations σ and $\tau \in \mathfrak{S}_m$ yield the ordering of the labels

$$\sigma(i_{k,l}) = j_{k,l}, \quad \text{and} \quad \tau^{-1}(j_{k,l}) = i_{k,l+1},$$
$$\text{with } i_{k,R_k+1} = i_{k,r} < i_{k,r},\ r = 2, R_k, \tag{6.20}$$

and the signs s, satisfying the constraints

$$s(i_{k,1}) = +1, \tag{6.21}$$

together with the signs t define the functions

$$\eta^{x\to y}_{k,l} = \begin{cases} 0 & \text{if } s(i_{k,l}) = t(j_{k,l}), \\ 1 & \text{if } s(i_{k,l}) = -t(j_{k,l}), \end{cases}$$

$$\eta^{y\to x}_{k,l} = \begin{cases} 0 & \text{if } t(j_{k,l}) = s(i_{k,l+1}), \\ 1 & \text{if } t(j_{k,l}) = -s(j_{k,l+1}) \end{cases}$$

and the operations

$$\zeta^x(i_{k,l}) = \begin{cases} T(\text{ranspose}) & \text{if } s(i_{k,l}) = -1, \\ I(\text{dentity}) & \text{if } s(i_{k,l}) = 1, \end{cases}$$

$$\zeta^y(j_{k,l}) = \begin{cases} T(\text{ranspose}) & \text{if } t(j_{k,l}) = -1, \\ I(\text{dentity}) & \text{if } t(j_{k,l}) = 1 \end{cases}$$

that perform the twisting of resolvents.

The structure of these functions is easily understood from the diagrams associated to π (equivalently, $[\omega]$). Each cycle in the diagram represents by a trace; to each dot is attached a resolvent if the dot is traversed clockwise by an arrow, and a transposed resolvent if it is counterclockwise; finally each change of orientation in the links corresponds to a J or a $\tilde{J}$ matrix.

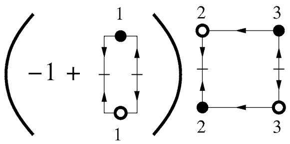
The functions are invariant under an independent global twist inside each trace, so the claim that our prescription depends only on the equivalence classes defined above is justified.

Finally the terms $\delta_{R_k,1}$ and $t(j)\delta_{R_k,1}$ in the definition of the functions are the analogues of the 1 and b appearing in $F_\pm$ in the example of Sect. 6.2. Here too they come only with the traces containing two resolvents, i.e. $R_k = 1$.

As an example the $[\{(1)(2)(3), (1)(23), (+,+,-), (-,-,+)\}]$ component of the basis for the orthogonal case would be

$$\begin{aligned} &\left(-1 + \operatorname{tr}\left(\frac{1}{x_1 - (X+T)} J \left(\frac{1}{y_1 - (Y+T^\dagger)}\right)^T J\right)\right) \\ &\quad \times \operatorname{tr}\left(\frac{1}{x_2 - (X+T)} J \left(\frac{1}{y_2 - (Y+T^\dagger)}\right)^T \left(\frac{1}{x_3 - (X+T)}\right)^T \right. \\ &\quad \left. \times J \frac{1}{y_3 - (Y+T^\dagger)}\right) \end{aligned}$$

which is represented by the diagram



There is a unified representation for the two bases of correlation functions corresponding to the orthogonal and symplectic cases. Define $N_{\mathcal{J}} = \sum_{k,l}(\eta_{k,l}^{x\to y} + \eta_{k,l}^{y\to x})$ (which is the total number of $\mathcal{J}$ matrices appearing in the traces), and $\Pi(s) = \prod_i^R s(i)$ for $s \in \mathbb{Z}_2^R$. Then we have[3]

$$\begin{aligned} &F_{\{\sigma,\tau,s,t\}}^{\mathcal{J}}(\{x\},\{y\},A,B) \\ &\quad = \frac{1}{n}\operatorname{tr}(\mathcal{J}^{N_{\mathcal{J}}})\Pi(s)\Pi(t)\prod_{k=1}^{p}\left(\delta_{R_k,1} + \operatorname{tr}\left(\prod_{l=1}^{R_k}\frac{1}{s(i_{k,l})x_{i_{k,l}} - A}\frac{1}{t(j_{k,l})y_{j_{k,l}} - B}\right)\right) \end{aligned}$$

[3]Here we have used (6.11) in order to remove the $\mathcal{J}$ and introduce the signs.

$$= \frac{1}{n}\operatorname{tr}(\mathcal{J}^{N_{\mathcal{J}}})\Pi(s)\Pi(t)(F^{U}_{\pi,e\pi e}(\{x\}_{2R},\{y\}_{2R},A,B))^{\frac{1}{2}} \tag{6.22}$$

where $\pi\in\mathfrak{S}_{2R}$ is the associated permutation following Appendix 5, e, $\{x\}_{2R}$ and $\{y\}_{2R}$ are also defined in that appendix, and $F^{U}_{\pi,\pi'}$ is the basis of correlation functions found in [12] for the unitary case. The sign can be computed through the limit $A,B\to\infty$. We prove the last equality in Appendix 5.

6.3.2 Recursion Relation

Using this basis we have the following theorem

Theorem 6.1 *The functions defined in* (6.18) *and* (6.19) *or equivalently in* (6.22) *satisfy the recursion relation*

$$\begin{aligned}
&\langle F^{\mathcal{J}}_{\{\sigma,\tau,s,t\}}(\{x\},\{y\},X+T,Y+T^{\dagger})\rangle_{(n)}\\
&\quad=\sum_{\{\sigma',\tau',s',t'\}}\overline{\mathcal{M}}_{(R)\{\sigma,\tau,s,t\}}^{\{\sigma',\tau',s',t'\}}(\{x\},\{y\},\alpha,\beta)\\
&\qquad\times\langle F^{\mathcal{J}}_{\{\sigma',\tau',s',t'\}}(\{x\},\{y\},\hat{X}+\hat{T},\hat{Y}+\hat{T}^{\dagger})\rangle_{(n-2)}
\end{aligned}\tag{6.23}$$

where α and β are the first eigenvalues of X and Y respectively, and $\hat{A}$ is the submatrix of size $n-2$ resulting from erasing the first and the last rows and columns of A, and where

$$\begin{aligned}
&\overline{\mathcal{M}}^{(R)\{\sigma',\tau',s',t'\}}_{\{\sigma,\tau,s,t\}}(\{x\},\{y\},\alpha,\beta)\\
&\quad=\left(\prod_{i=1}^{R}\left(\delta_{\sigma(i),\sigma'(i)}\delta_{s(i),s'(i)}\delta_{t(\pi(i)),t'(\pi(i))}+\frac{1}{s(i)x_i+\alpha}\frac{1}{t(\sigma(i))y_{\sigma(i)}+\beta}\right)\right)\\
&\quad=\left(\prod_{i=1}^{R}\left(\delta_{\tau(i),\tau'(i)}\delta_{s(i),s'(i)}\delta_{t(\tau(i)),t'(\tau(i))}+\frac{1}{s(i)x_i-\alpha}\frac{1}{t(\tau(i))y_{\tau(i)}-\beta}\right)\right).
\end{aligned}$$

Proof The proof of this theorem is given in Appendix 4. □

The first thing to notice is that (see Appendix 5 for a proof) the matrix $\overline{\mathcal{M}}$ is again closely related to the corresponding recursion matrix $\mathcal{M}$ found in [12] for the unitary case. Recall that

$$\mathcal{M}^{(2R)}_{\pi,\pi'}(\{x\}\{y\},\xi,\eta)=\prod_{i=1}^{R}\left(\delta_{\pi(i),\pi'(i)}+\frac{1}{x_i-\xi}\frac{1}{y_{\pi(i)}-\eta}\right), \tag{6.24}$$

then taking into account the bijection defined in Appendix 5 we find that

$$\overline{\mathcal{M}}^{(R)\{\tau,\tau',t,t'\}}_{\{\sigma,\sigma',s,s'\}}(\{x\}_R,\{y\}_R,\alpha,\beta)=\mathcal{M}^{(2R)}_{\pi,\pi'}(\{x\}_{2R},\{y\}_{2R},-\alpha,-\beta) \tag{6.25}$$

which again connects the orthogonal and symplectic cases with the unitary case. The precise relation between the arguments of the two sides of this equation is defined in Appendix 5. Trivial consequences of this fact are the commutativity property of $\overline{\mathcal{M}}$

$$[\overline{\mathcal{M}}^{(R)}(\{x\},\{y\},\alpha,\beta),\overline{\mathcal{M}}^{(R)}(\{x\},\{y\},\xi,\eta)]=0, \tag{6.26}$$

and the symmetry $\mathcal{M}=\mathcal{M}^T$.

The recursion relation we just found is valid for any value of n such that $n\geq 3$. The special cases $n=2$ and $n=1$ correspond to the initial condition for the recursion relation in the even n and odd n case respectively.

6.3.3 Initial Conditions

Let us consider first the case n even. For any $n>2$ even, the recursion relation is valid. The last step for $n=2$ requires a slightly more careful analysis. In a 2×2 strictly upper triangular matrix, the only term is $T_{1,2}$. Since the "last column/first row" integration reduces to that of $T_{1,2}$, the procedure explained in Appendix 4 is still valid, and the recursion relation can be naively applied just by considering that when $n=0$, all the traces are equal to zero in the correlation functions (or equivalently taking the strict limit where x_i, $y_i\to\infty$ for $i=1,\dots,R$). This gives us the initial condition vector

$$(I_0^{\mathcal{J}})_{\{\sigma,\tau,s,t\}}=\mathrm{tr}(\mathcal{J}^{N_J})\Pi(s)\Pi(t)\delta_{\sigma,\tau}. \tag{6.27}$$

Explicitly, the two cases $\mathcal{J}=J$ and $\mathcal{J}=\tilde{J}$ are

$$\begin{aligned}(I_0^{J})_{\{\sigma,\tau,s,t\}}&=\Pi(s)\Pi(t)\delta_{\sigma,\tau},\\ (I_0^{\tilde{J}})_{\{\sigma,\tau,s,t\}}&=\delta_{\sigma,\tau}.\end{aligned} \tag{6.28}$$

The case $n=1$ corresponds to the case where there is no triangular matrix at all, thus no integration. The initial condition corresponds in this case to the vector

$$(I_1^{J})_{\{\sigma,\tau,s,t\}}(\{x\},\{y\})=\mathrm{tr}(\mathcal{J}^{N_J})\prod_{k=1}^{p}\left(t(j_{k,l})\delta_{R_k,1}+\prod_{l=1}^{R_k}\frac{1}{x_{i_{k,l}}y_{j_{k,l}}}\right). \tag{6.29}$$

7 Correlation Functions over O(n) and Sp($2m$): The Final Expression

In this section we find a determinantal formula for the correlation functions in Sects. 4 and 5. We use need the matrix determinant Mdet defined as,

$$\mathrm{Mdet}(M)=\sum_{\sigma\in\mathfrak{S}(n)}(-1)^{\sigma}\prod_{i=1}^{n}M_{i,\sigma(i)} \tag{7.1}$$

where each $M_{i,j}$ is a matrix. This means that $\mathrm{Mdet}(M)$ is itself a matrix. The ordering of matrices in the product does not matter if the matrices commute with one another.

- **The** O($2m$) **case**

Take Theorem 4.3 for $n=2m$ and the recursion relation found in Sect. 6. After some algebra we find

$$\begin{aligned}&\frac{\int_{\mathrm{O}(2m)}\mathrm{d}O\,F^{(R)}(X,OYO^T)\mathrm{e}^{-\gamma\,\mathrm{tr}(XOYO^T)}}{\int_{\mathrm{O}(2m)}\mathrm{d}O\,\mathrm{e}^{-\gamma\,\mathrm{tr}(XOYO^T)}}\\ &\quad=\frac{\mathrm{Mdet}(\mathrm{e}^{2\gamma X_kY_j}\overline{\mathcal{M}}^{(R)}(\{x\},\{y\},iX_k,iY_j)+\mathrm{e}^{-2\gamma X_kY_j}\overline{\mathcal{M}}^{(R)}(\{x\},\{y\},iX_k,-iY_j))_{k,j=1,\dots,m}}{\det(2\cosh(2\gamma X_kY_j))_{k,j=1,\dots,m}}I_0^J\end{aligned}$$

$$= \sum_{t\in\mathbb{Z}_2^m} \frac{\mathrm{Mdet}(\mathrm{e}^{2\gamma t_j X_k Y_j}\overline{\mathcal{M}}^{(R)}(\{x\},\{y\},iX_k,i\,t_jY_j))_{k,j=1,\dots,m}}{\det(2\cosh(2\gamma X_kY_j))_{k,j=1,\dots,m}} I_0^J \tag{7.2}$$

where $F^{(R)}$ is a vector with $(2R)!$ components, Mdet is a $(2R)!\times(2R)!$ matrix and I_0^J is the vector defined in (6.28).

- **The** $\mathrm{O}(2m+1)$ **case**

Take now the $n=2m+1$ part of Theorem 4.3. With the results of Sect. 6 we find

$$\frac{\int_{\mathrm{O}(2m+1)} \mathrm{d}O\, F^{(R)}(X,OYO^T)\mathrm{e}^{-\gamma\,\mathrm{tr}(XOYO^T)}}{\int_{\mathrm{O}(2m+1)} \mathrm{d}O\mathrm{e}^{-\gamma\,\mathrm{tr}(XOYO^T)}}$$

$$= \frac{\mathrm{Mdet}(\mathrm{e}^{2\gamma X_kY_j}\overline{\mathcal{M}}^{(R)}(\{x\},\{y\},iX_k,iY_j)-\mathrm{e}^{-2\gamma X_kY_j}\overline{\mathcal{M}}^{(R)}(\{x\},\{y\},iX_k,-iY_j))_{k,j=1,\dots,m}}{\det(2\sinh(2\gamma X_kY_j))_{k,j=1,\dots,m}} I_1^J(\{x\},\{y\})$$

$$= \sum_{t\in\mathbb{Z}_2^m}\left(\prod_{l=1}^m t_l\right)\frac{\mathrm{Mdet}(\mathrm{e}^{2\gamma t_j X_k Y_j}\overline{\mathcal{M}}^{(R)}(\{x\},\{y\},iX_k,it_jY_j))_{k,j=1,\dots,m}}{\det(2\sinh(2\gamma X_kY_j))_{k,j=1,\dots,m}} I_1^J(\{x\},\{y\}) \tag{7.3}$$

where I_1^J is given in (6.29).

- **The** $\mathrm{Sp}(2m)$ **case**

Finally take Theorem 5.3 and the corresponding part of Sect. 6. After some algebra

$$\frac{\int_{\mathrm{Sp}(2m)} \mathrm{d}S\, F^{(R)}(X,SYS^\dagger)\mathrm{e}^{-\gamma\,\mathrm{tr}(XSYS^\dagger)}}{\int_{\mathrm{Sp}(2m)} \mathrm{d}S\mathrm{e}^{-\gamma\,\mathrm{tr}(XSYS^\dagger)}}$$

$$= \frac{\mathrm{Mdet}(\mathrm{e}^{2\gamma X_kY_j}\overline{\mathcal{M}}^{(R)}(\{x\},\{y\},iX_k,iY_j)-\mathrm{e}^{-2\gamma X_kY_j}\overline{\mathcal{M}}^{(R)}(\{x\},\{y\},iX_k,-iY_j))_{k,j=1,\dots,m}}{\det(2\sinh(2\gamma X_kY_j))_{k,j=1,\dots,m}} I_0^{\tilde{J}}$$

$$= \sum_{t\in\mathbb{Z}_2^m}\left(\prod_{l=1}^m t_l\right)\frac{\mathrm{Mdet}(\mathrm{e}^{2\gamma t_j X_k Y_j}\overline{\mathcal{M}}^{(R)}(\{x\},\{y\},iX_k,it_jY_j))_{k,j=1,\dots,m}}{\det(2\sinh(2\gamma X_kY_j))_{k,j=1,\dots,m}} I_0^{\tilde{J}} \tag{7.4}$$

where $I_0^{\tilde{J}}$ is given in (6.28).

8 Concluding Remarks

8.1 A Remark on Contour Deformation in Matrix Integrals

We first return to the transformation of our original integrals over two real antisymmetric, resp. two antiselfdual real quaternionic, matrices into integrals over J-, resp. $\tilde{J}$-, antisymmetric complex matrices. Define the following measure on $M_n(\mathbb{C})\times M_n(\mathbb{C})$, the set of pairs of two complex $n\times n$ matrices,

$$\mathrm{e}^{-\mathrm{tr}(\frac{\alpha_1}{2}M_1^2+\frac{\alpha_2}{2}M_2^2+\gamma M_1M_2)}\mathrm{d}M_1\mathrm{d}M_2. \tag{8.1}$$

We will consider two hyperplanes of $M_n(\mathbb{C})\times M_n(\mathbb{C})$ and the measure on these hyperplanes induced by the measure above.

The first hyperplane is defined by the equations

$$\left.\begin{aligned} M_i &= M_i^* \equiv A_i \\ A_i &= -A_i^T \end{aligned}\right\}, \quad i = 1, 2 \tag{8.2}$$

endowed with the induced measure (3.1) reproduces the two real antisymmetric matrix integral of Sect. 3.1.1. The second hyperplane, which describes one complex J-antisymmetric matrix, i.e. $J\mathcal{A}_n(\mathbb{C})$, is defined by the equations

$$\begin{aligned} M_1 &= M_2^\dagger \equiv M, \\ JM &= -M^T J \end{aligned} \tag{8.3}$$

with the induced measure (3.4).

In this construction, the real antisymmetric two matrix integrals and the complex J-antisymmetric matrix integrals are nothing but the same integral on different hyperplanes of $M_n(\mathbb{C}) \times M_n(\mathbb{C})$. By the counting done above, these two hyperplanes have the same dimension, and a plausible interpretation of Theorem 3.1 is that it results from a contour deformation taking the first set of matrix integrals into the second one.

Similarly we can consider two different hyperplanes in the space of pairs of quaternionic matrices, namely the one defining $Q\mathcal{A}_m \times Q\mathcal{A}_m$

$$\left.\begin{aligned} M_i &= M_i^* \equiv Q_i \\ Q_i &= -Q_i{}^\dagger \end{aligned}\right\}, \quad i = 1, 2 \tag{8.4}$$

with the induced measure (3.7), and the one defining $\tilde{J}\mathcal{A}_{2m}$

$$\begin{aligned} M_1 &= M_2^\dagger \equiv M, \\ \tilde{J}M &= -M^T \tilde{J} \end{aligned} \tag{8.5}$$

with the induced measure (3.13). Both hyperplanes have the same dimension and we may again interpret Theorem 3.2 as resulting from a contour deformation.

This apparently much simpler and intuitive approach has the drawback of neglecting convergence issues. This is the reason why we chose to prove our results by use of loop equations.

Other choices of deformations to other hyperplanes are also conceivable.

8.2 Comparison with the Duistermaat–Heckman Form

First observe that we may rewrite the main Theorems 4.3 and 5.3 together with Theorem 4-1 in [12] into a unified form involving an integration over a set T_G of complex triangular matrices, as already mentioned in the Introduction. When G is $O(n)$, $Sp(2m)$ or $U(n)$, T_G corresponds respectively to the set of J, $\tilde{J}$-antisymmetric, or unconstrained, strictly upper triangular matrices. Moreover, we notice that in each case, the set T_G is precisely the derived ideal $[\mathfrak{b}, \mathfrak{b}] =: \mathfrak{n}_+$ of the Borel subalgebra $\mathfrak{b}$ associated with the choice of Cartan algebra made above. This is in fact the space generated by the positive roots $\mathfrak{n}_+ = \oplus_{\alpha>0}\, \mathfrak{g}^\alpha$.

It is thus natural to conjecture that an analogous formula holds for any compact group, with the identification of T_G with the subalgebra $\mathfrak{n}_+$:

Conjecture 8.1 *For any compact group G*

$$\int_G \mathrm{d}\Omega\, F(X^a, \Omega Y^a \Omega^{-1}) \mathrm{e}^{-\operatorname{tr}(X^a \Omega Y^a \Omega^{-1})}$$
$$= c \sum_{w \in \mathcal{W}} \frac{\mathrm{e}^{+\operatorname{tr}(X w(Y))}}{\prod_{\alpha>0} \alpha(X)\alpha(w(Y))}$$
$$\times \int_{\mathfrak{n}_+ = [\mathfrak{b},\mathfrak{b}]} \mathrm{d}T\, F(iX + T, i w(Y) + T^\dagger) \mathrm{e}^{-\operatorname{tr}(T T^\dagger)}. \tag{8.6}$$

This form of our result has to be confronted with the form given by Duistermaat–Heckman's localization theorem, (2.4). *Note that the integration over* $\mathfrak{n}^+$ *plays here the role played in Sect.* 2.2 *by the integration over the "fluctuations"* $A \in \mathfrak{g}\backslash\mathfrak{h}$ *in* (2.4). *This points to a possible much more compact and geometric derivation of our results.*

8.3 Other Comments

First, it is remarkable that the recursion on n for triangular integrals involves the *same* matrix $\mathcal{M}$ for all cases U(n), O(n), Sp($2n$). Only the initial conditions differ. This fact needs to be understood, and it shows that the matrix $\mathcal{M}$ is universal. Moreover, its commutation properties suggest the existence of some underlying integrable structure. The symmetries of the group under consideration are reflected in the symmetries of the spectral parameters at which $\mathcal{M}$ is evaluated. The initial conditions also seem to have such symmetries, and it is remarkable that those symmetries are reminiscent of root lattices of size R or $2R$ (we started with a root lattice of size n or $2n$). This suggests a duality between R and n, similar to that of the "supersymmetric" method of evaluation of determinantal correlation functions [23]. Remarkably, the triangular matrix ensembles as those considered here seem to play an important role in generalizations of the so-called Razumov–Stroganov conjecture. Indeed multidegrees of the corresponding matrix varieties are solutions of the quantum Knizhnik–Zamolodchikov equation based on the root systems of type A, B, C and D [24], thus pointing again towards some possible integrable structure. It might be also interesting to attack the "angular" integrals considered in this paper with character expansion techniques, see [25, 26] for recent references.

Our last comment is that it would be highly desirable to know how to compute integrals like (1.1) on other orbits. For example, little is known about the integral over the O(n) group when X and Y are *symmetric* real matrices (see however [27]).

Acknowledgements We would like to thank M. Bauer and M. Talon for very helpful discussions. This work was supported by the Enigma European RTN network under contract MRT-CT-2004-5652, and partly supported by the ANR project Géométrie et intégrabilité en physique mathématique ANR-05-BLAN-0029-01, and by the Enrage European network MRTN-CT-2004-005616. B.E. thanks the CRM (Montreal QC) for its support. J.-B.Z. thanks KITP, Santa Barbara, for hospitality and support, where part of this work was carried out, with partial support by the National Science Foundation under Grant No. PHY99-07949.

Appendix 1: Quaternions

We review here some well known facts about quaternions to fix our notations, which follow basically those of [19]. We shall only consider the set of *real quaternions*, which is the algebra over $\mathbb{R}$ generated by 4 elements: the neutral element e_0, which by an abuse of notation we often write $e_0 = 1$, and e_i, $i = 1, 2, 3$,

$$q = q^{(0)}e_0 + q^{(1)}e_1 + q^{(2)}e_2 + q^{(3)}e_3, \quad q^{(\alpha)} \in \mathbb{R} \tag{9.1}$$

with multiplication $e_i^2 = e_1e_2e_3 = -1$, from which it follows that $e_1e_2 = -e_2e_1 = e_3$ and its cyclic permutations. One may represent e_0 by Id_2 the 2×2 identity matrix, and the e_i in terms of 2×2 Pauli matrices by

$$e_i = -i\sigma_i. \tag{9.2}$$

The conjugate quaternion of q is defined as

$$\bar{q} = q^{(0)}1 - q^{(1)}e_1 - q^{(2)}e_2 - q^{(3)}e_3.$$

(This is also called *hermitian conjugate*, which is justified by the fact that Pauli matrices are hermitian.) Note that $q\bar{q} := |q|^2 = |q^{(0)}|^2 + |q^{(1)}|^2 + |q^{(2)}|^2 + |q^{(3)}|^2$, the square norm of the quaternion, and hence $q \neq 0$ has an inverse $q^{-1} = \bar{q}/|q|^2$. Real quaternions form a non-commutative field. Note also that conjugation reverses the order of factors of a product $\overline{(q_1q_2)} = \bar{q}_2\bar{q}_1$.

Quaternionic Matrices

We now consider matrices Q with real quaternionic elements Q_{ij}, $i, j = 1, \dots, m$. Alternatively, using (9.2), one may regard also Q as a $2m \times 2m$ matrix with 2×2 blocks made of real combinations of Id_2 and the Pauli matrices. One may apply to Q the same conjugation as defined above. One may also transpose Q. The *dual* Q^R of a quaternionic matrix Q is the matrix

$$(Q^R)_{ij} = \bar{Q}_{ji}. \tag{9.3}$$

(This is also the hermitian conjugate $Q^\dagger$ of Q in the usual sense.) A real quaternionic matrix is thus *self-dual* if

$$Q^R = Q = Q^\dagger = (Q_{ij}) = (\bar{Q}_{ji}). \tag{9.4}$$

A real quaternionic matrix is *anti self-dual* if

$$Q^R = Q^\dagger = -Q; \tag{9.5}$$

it is thus anti-hermitian. In particular, its diagonal matrix elements are such that $Q_{ii}^{(0)} = 0$.

On quaternionic matrices, we may define the ordinary trace

$$\mathrm{tr}(Q) = \sum_{i=1}^{m} Q_{ii},$$

which is in general a quaternion, or

$$\mathrm{tr}_0(Q) = 2\sum_{i=1}^{m} Q_{ii}^{(0)} = \mathrm{tr}(Q) + \overline{\mathrm{tr}(Q)} \tag{9.6}$$

which is a scalar. Note that $\mathrm{tr}_0(Q)$ is nothing else than the trace of the corresponding $2m \times 2m$ matrix.

Symplectic Group Sp$(2m)$

Consider the $2m \times 2m$ matrix

$$\mathcal{A} = \begin{pmatrix} 0 & \mathbf{1}_m \\ -\mathbf{1}_m & 0 \end{pmatrix} \tag{9.7}$$

and the associated skew-symmetric bilinear form

$$(X, Y) = X^T \mathcal{A} Y = \sum_{i=1}^{m} (x_i y_{i+m} - y_i x_{i+m}). \tag{9.8}$$

The symplectic group Sp$(2m)$ is the group of $2m \times 2m$ matrices leaving this form invariant

$$S^T \mathcal{A} S = \mathcal{A}. \tag{9.9}$$

In the basis where $X^T = (x_1, x_{m+1}, x_2, x_{m+2}, \ldots)$, the matrix

$$\mathcal{A} = \mathrm{diag}\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\},$$

and the symplectic group is generated by real quaternionic unitary matrices $S^R = S^\dagger = S^{-1}$.

Appendix 2: Loop equations I

In this appendix we show how to compute loop equations in the real antisymmetric two matrix integral and in the J-antisymmetric complex matrix integral.

10.1 Loop Equations for the 2 Real Antisymmetric Matrix Integral

10.1.1 Loop Equations

Schwinger–Dyson equations, also called loop equations in the case of matrix integrals, merely amount to saying that the integral of a total derivative vanishes:

$$0 = \sum_{i<j} \int dA_1 dA_2 \frac{\partial}{\partial A_{1ij}} (f(A_1, A_2)_{ij}\, \mathrm{e}^{-\mathrm{tr}(\frac{\alpha_1}{2} A_1^2 + \frac{\alpha_2}{2} A_2^2 + \gamma A_1 A_2)}) \tag{10.1}$$

where $f(A_1, A_2) = -f^t(A_1, A_2)$ is any sufficiently regular matrix valued function; in particular f can be any non-commutative polynomial in A_1 and A_2, and may contain also product of traces of polynomials.

The loop equation thus turns into an equality between expectation values:

$$\langle K_1(f)\rangle = \langle \mathrm{tr}((\alpha_1 A_1 + \gamma A_2) f(A_1, A_2))\rangle \tag{10.2}$$

where

$$K_1(f) = \sum_{i<j} \frac{\partial f(A_1, A_2)_{ij}}{\partial A_{1ij}}. \tag{10.3}$$

Notice that $K_1(f)$ is linear and satisfies Leibniz rule:

$$K_1(fg) = K_1(f)g + fK_1(g). \tag{10.4}$$

The most general f we shall consider is of the type:

$$f(A_1, A_2) = \frac{1}{2}(F_0(A_1, A_2) - F_0(A_1, A_2)^T)\prod_{r=1}^{R}\operatorname{tr}(F_r(A_1, A_2)) \tag{10.5}$$

where F_0 is an odd degree non commutative monomial of A_1 and A_2:

$$F_0(A_1, A_2) = A_2^{l_{0,0}} A_1^{k_{0,1}} A_2^{l_{0,1}} A_1^{k_{0,2}} \dots A_2^{l_{0,p_0-1}} A_1^{k_{0,p_0}} A_2^{l_{0,p_0}} \tag{10.6}$$

and each F_r with $r \geq 1$ is an even degree non commutative monomial:

$$F_r(A_1, A_2) = A_1^{k_{r,1}} A_2^{l_{r,1}} A_1^{k_{r,2}} \dots A_2^{l_{r,p_r-1}} A_1^{k_{r,p_r}} A_2^{l_{r,p_r}} \tag{10.7}$$

and we call $\deg(f)$ the total number of matrices A_1 + the total number of matrices A_2.

Then compute:

$$\begin{aligned}
&K_1(F_0 - F_0^T) \\
&= \sum_{q=1}^{p_0}\sum_{m=0}^{k_{0,q}-1}\sum_{i<j}[(A_2^{l_{0,0}} A_1^{k_{0,1}} A_2^{l_{0,1}} \dots A_2^{l_{q-1}} A_1^{m})_{ii}(A_1^{k_{0,q}-m-1} A_2^{l_{0,q}} \dots A_2^{l_{0,p_0}})_{jj} \\
&\quad - (A_2^{l_{0,0}} A_1^{k_{0,1}} A_2^{l_{0,1}} \dots A_2^{l_{q-1}} A_1^{m})_{ij}(A_1^{k_{0,q}-m-1} A_2^{l_{0,q}} \dots A_2^{l_{0,p_0}})_{ij} \\
&\quad - (A_2^{l_{0,0}} A_1^{k_{0,1}} A_2^{l_{0,1}} \dots A_2^{l_{q-1}} A_1^{m})_{ji}(A_1^{k_{0,q}-m-1} A_2^{l_{0,q}} \dots A_2^{l_{0,p_0}})_{ji} \\
&\quad + (A_2^{l_{0,0}} A_1^{k_{0,1}} A_2^{l_{0,1}} \dots A_2^{l_{q-1}} A_1^{m})_{jj}(A_1^{k_{0,q}-m-1} A_2^{l_{0,q}} \dots A_2^{l_{0,p_0}})_{ii}] \\
&= \sum_{q=1}^{p_0}\sum_{m=0}^{k_{0,q}-1}[\operatorname{tr}(A_2^{l_{0,0}} A_1^{k_{0,1}} A_2^{l_{0,1}} A_1^{k_{0,2}} \dots A_2^{l_{0,q-1}} A_1^{m})\operatorname{tr}(A_1^{k_{0,q}-m-1} A_2^{l_{0,q}} \dots A_2^{l_{0,p_0}}) \\
&\quad - (-1)^{k_{0,1}+\dots+k_{0,q-1}+m+l_{0,0}+\dots+l_{0,q-1}} \\
&\quad \times \operatorname{tr}(A_1^{m} A_2^{l_{0,q-1}} \dots A_1^{k_{0,2}} A_2^{l_{0,1}} A_1^{k_{0,1}} A_2^{l_{0,0}} A_1^{k_{0,q}-m-1} A_2^{l_{0,q}} \dots A_2^{l_{0,p_0}})].
\end{aligned} \tag{10.8}$$

This equality is known as the split rule.

Then we have for any antisymmetric matrix C:

$$\begin{aligned}
&K_1(C\operatorname{tr}(F_r)) \\
&= \sum_{q=1}^{p_r}\sum_{m=0}^{k_{r,q}-1}\sum_{i<j}\sum_{s}[(A_1^{k_{r,1}} A_2^{l_{r,1}} A_1^{k_{r,2}} \dots A_2^{l_{r,q-1}} A_1^{m})_{si} C_{ij} (A_1^{k_{r,q}-m-1} A_2^{l_{r,q}} \dots A_2^{l_{r,p_r}})_{js} \\
&\quad - (A_1^{k_{r,1}} A_2^{l_{r,1}} A_1^{k_{r,2}} \dots A_2^{l_{r,q-1}} A_1^{m})_{sj} C_{ij} (A_1^{k_{r,q}-m-1} A_2^{l_{r,q}} \dots A_2^{l_{r,p_r}})_{is}] \\
&= \sum_{q=1}^{p_r}\sum_{m=0}^{k_{r,q}-1}\operatorname{tr}(A_1^{k_{r,1}} A_2^{l_{r,1}} A_1^{k_{r,2}} \dots A_2^{l_{r,q-1}} A_1^{m} C A_1^{k_{r,q}-m-1} A_2^{l_{r,q}} \dots A_2^{l_{r,p_r}}).
\end{aligned} \tag{10.9}$$

This equality is known as the merge rule.

Due to Leibniz rule and using repeatedly the split and merge rules, we find that if f has the form of (10.5) then $K_1(f)$ is a linear combination of monomial invariant functions of degree $\leq \deg(f) - 1$.

The loop equations read:

$$\begin{aligned} \langle \operatorname{tr}(\alpha_1 A_1 f + \gamma A_2 f)\rangle &= K_1(f), \\ \langle \operatorname{tr}(\alpha_2 A_2 f + \gamma A_1 f)\rangle &= K_2(f) \end{aligned} \tag{10.10}$$

or equivalently:

$$\begin{aligned} \langle \operatorname{tr}(A_1 f)\rangle &= \frac{\alpha_2}{\delta} K_1(f) - \frac{\gamma}{\delta} K_2(f), \\ \langle \operatorname{tr}(A_2 f)\rangle &= \frac{\alpha_1}{\delta} K_2(f) - \frac{\gamma}{\delta} K_1(f). \end{aligned} \tag{10.11}$$

10.1.2 Polynomial Invariant Functions

$F(A_1, A_2)$ is a monomial invariant function of two antisymmetric matrices A_1, A_2, if it is either:

$$\begin{cases} F = 1 \quad \text{or} \\ F(A_1, A_2) = \operatorname{tr}(A_1 f(A_1, A_2)) \quad \text{or} \\ F(A_1, A_2) = \operatorname{tr}(A_2 f(A_1, A_2)) \end{cases} \tag{10.12}$$

where f is of the following form:

$$f(A_1, A_2) = F_0(A_1, A_2) \prod_{r=1}^{R} \operatorname{tr}(F_r(A_1, A_2)) \tag{10.13}$$

where F_0 is an odd degree non commutative monomial of A_1 and A_2:

$$F_0(A_1, A_2) = A_2^{l_{0,0}} A_1^{k_{0,1}} A_2^{l_{0,1}} A_1^{k_{0,2}} \ldots A_2^{l_{0,p_0-1}} A_1^{k_{0,p_0}} A_2^{l_{0,p_0}} \tag{10.14}$$

and each F_r with $r \geq 1$ is an even degree non commutative monomial:

$$F_r(A_1, A_2) = A_1^{k_{r,1}} A_2^{l_{r,1}} A_1^{k_{r,2}} \ldots A_2^{l_{r,p_r-1}} A_1^{k_{r,p_r}} A_2^{l_{r,p_r}} \tag{10.15}$$

and we call $\deg(F)$ the total number of matrices A_1 + the total number of matrices A_2. Notice also that f can be antisymmetrized without changing F, and thus f can be taken of the form of (10.5).

Notice that if $\deg(F)$ is odd, we have:

$$\langle F \rangle = 0 \tag{10.16}$$

If $F = 1$, i.e. if $\deg(F) = 0$ we have:

$$\langle 1 \rangle = 1 \tag{10.17}$$

and if $\deg(F) > 0$, and $F = \operatorname{tr}(A_1 f)$, the loop equations imply:

$$\langle F \rangle = \frac{\alpha_2}{\delta} K_1(f) - \frac{\gamma}{\delta} K_2(f) \tag{10.18}$$

where the right hand side is the expectation value of a polynomial invariant function of degree $\leq \deg(F)-2$. And if $\deg(F)>0$, and $F=\mathrm{tr}(A_2 f)$, the loop equations imply:

$$\langle F\rangle = \frac{\alpha_1}{\delta}K_2(f) - \frac{\gamma}{\delta}K_1(f) \tag{10.19}$$

where again the right hand side is the expectation value of a polynomial invariant function of degree $\leq \deg(F)-2$.

In other words, the loop equations allow to compute every expectation of polynomial invariant functions by recursion on the degree.

Notice also that the expectation value of any monic monomial invariant function is a polynomial in $\frac{\alpha_1}{\delta}$, $\frac{\alpha_2}{\delta}$ and $\frac{\gamma}{\delta}$.

10.2 Loop Equations for the Complex J-Antisymmetric Matrix Integral

10.2.1 Loop Equations

Similarly to the previous section, loop equations, in the case of a complex J-antisymmetric matrix integral, can be written:

$$\begin{aligned} 0 = \sum_{i<j}\int dM\left(\frac{\partial}{\partial\,\mathrm{Re}\,M_{i,n+1-j}} - i\frac{\partial}{\partial\,\mathrm{Im}\,M_{i,n+1-j}}\right)& \\ \times\,(f(M,M^\dagger)_{i,n+1-j}\mathrm{e}^{-\mathrm{tr}(\frac{\alpha_1}{2}M^2+\frac{\alpha_2}{2}M^{\dagger 2}+\gamma MM^\dagger)})& \end{aligned} \tag{10.20}$$

where $f(M,M^\dagger) = -Jf(M,M^\dagger)^T J$ is any sufficiently regular matrix valued function, in particular f can be any non-commutative polynomial in M and $M^\dagger$, and may contain also product of traces of polynomials.

The loop equation thus turns into an equality between expectation values:

$$\langle K_1(f)\rangle = \langle \mathrm{tr}((\alpha_1 M+\gamma M^\dagger)f(M,M^\dagger))\rangle \tag{10.21}$$

where

$$K_1(f) = \frac{1}{2}\sum_{i<j}\left(\frac{\partial}{\partial\,\mathrm{Re}\,M_{i,n+1-j}} - i\frac{\partial}{\partial\,\mathrm{Im}\,M_{i,n+1-j}}\right)f_{i,n+1-j}. \tag{10.22}$$

Notice that $K_1(f)$ is linear and satisfies Leibniz rule:

$$K_1(fg) = K_1(f)g + fK_1(g). \tag{10.23}$$

The most general f we shall consider is of the type:

$$f(M,M^\dagger) = \frac{1}{2}(F_0(M,M^\dagger) - JF_0(M,M^\dagger)^T J)\prod_{r=1}^{R}\mathrm{tr}(F_r(M,M^\dagger)) \tag{10.24}$$

where $F_0(M,M^\dagger)$ is an odd degree non commutative monomial of M and $M^\dagger$:

$$F_0(A_1,A_2) = M^{\dagger l_{0,0}}M^{k_{0,1}}M^{\dagger l_{0,1}}M^{k_{0,2}}\dots M^{\dagger l_{0,p_0-1}}M^{k_{0,p_0}}M^{\dagger l_{0,p_0}} \tag{10.25}$$

and each $F_r(M,M^\dagger)$ with $r\geq 1$ is an even degree non commutative monomial:

$$F_r(M,M^\dagger) = M^{k_{r,1}}M^{\dagger l_{r,1}}M^{k_{r,2}}\dots M^{\dagger l_{r,p_r-1}}M^{k_{r,p_r}}M^{\dagger l_{r,p_r}} \tag{10.26}$$

and we call $\deg(f)$ the total number of matrices M + the total number of matrices $M^\dagger$.

We have

$$\begin{aligned}
&K_1(F_0 - JF_0^T J)\\
&\quad= \sum_{q=1}^{p_0}\sum_{m=0}^{k_{0,q}-1}\sum_{i<j}[(M^{\dagger l_{0,0}}M^{k_{0,1}}\dots M^{\dagger l_{0,q-1}}M^m)_{i,i}\\
&\qquad\times(M^{k_{0,q}-m-1}M^{\dagger l_{0,q}}\dots M^{k_{0,p_0}}M^{\dagger l_{0,p_0}})_{n+1-j,n+1-j}\\
&\qquad-(M^{\dagger l_{0,0}}M^{k_{0,1}}\dots M^{\dagger l_{0,q-1}}M^m)_{i,j}(M^{k_{0,q}-m-1}M^{\dagger l_{0,q}}\dots M^{k_{0,p_0}}M^{\dagger l_{0,p_0}})_{n+1-i,n+1-j}\\
&\qquad-(M^{\dagger l_{0,0}}M^{k_{0,1}}\dots M^{\dagger l_{0,q-1}}M^m)_{j,i}(M^{k_{0,q}-m-1}M^{\dagger l_{0,q}}\dots M^{k_{0,p_0}}M^{\dagger l_{0,p_0}})_{n+1-j,n+1-i}\\
&\qquad+(M^{\dagger l_{0,0}}M^{k_{0,1}}\dots M^{\dagger l_{0,q-1}}M^m)_{j,j}(M^{k_{0,q}-m-1}M^{\dagger l_{0,q}}\dots M^{k_{0,p_0}}M^{\dagger l_{0,p_0}})_{n+1-i,n+1-i}]\\
&\quad= \sum_{q=1}^{p_0}\sum_{m=0}^{k_{0,q}-1}[\mathrm{tr}(M^{\dagger l_{0,0}}M^{k_{0,1}}\dots M^{\dagger l_{0,q-1}}M^m)\\
&\qquad\times\mathrm{tr}(M^{k_{0,q}-m-1}M^{\dagger l_{0,q}}\dots M^{k_{0,p_0}}M^{\dagger l_{0,p_0}})\\
&\qquad-(-1)^{l_{0,0}+\dots+l_{0,q-1}+k_{0,1}+\dots+k_{0,q-1}+m}\\
&\qquad\times\mathrm{tr}(M^m M^{\dagger l_{0,q-1}}\dots M^{k_{0,1}}M^{\dagger l_{0,0}}M^{k_{0,q}-m-1}M^{\dagger l_{0,q}}\dots M^{k_{0,p_0}}M^{\dagger l_{0,p_0}})].
\end{aligned} \tag{10.27}$$

Notice that the split rule for J-antisymmetric complex matrices is identical to the split rule for real antisymmetric matrices equation (10.8).

Similarly, we compute the merge rule (where $C = -JC^T J$ is any J-antisymmetric complex matrix):

$$\begin{aligned}
&K_1(C\,\mathrm{tr}(F_r))\\
&\quad= \sum_{q=1}^{p_r}\sum_{m=0}^{k_{0,q}-1}\sum_{i<j}\sum_s[(M^{k_{r,1}}\dots M^{\dagger l_{r,q-1}}M^m)_{s,i}\\
&\qquad\times C_{i,n+1-j}(M^{k_{r,q}-m-1}M^{\dagger l_{r,q}}\dots M^{k_{r,p_r}}M^{\dagger l_{r,p_r}})_{n+1-j,s}\\
&\qquad-(M^{k_{r,1}}\dots M^{\dagger l_{r,q-1}}M^m)_{s,j}\\
&\qquad\times C_{i,n+1-j}(M^{k_{r,q}-m-1}M^{\dagger l_{r,q}}\dots M^{k_{r,p_r}}M^{\dagger l_{r,p_r}})_{n+1-i,s}]\\
&\quad= \sum_{q=1}^{p_r}\sum_{m=0}^{k_{0,q}-1}\mathrm{tr}(M^{k_{r,1}}\dots M^{\dagger l_{r,q-1}}M^m C M^{k_{r,q}-m-1}M^{\dagger l_{r,q}}\dots M^{k_{r,p_r}}M^{\dagger l_{r,p_r}}).
\end{aligned} \tag{10.28}$$

And again the merge rule for J-antisymmetric complex matrices is identical to the merge rule for real antisymmetric matrices equation (10.9).

We conclude that the expectation values of invariant polynomials of M and $M^\dagger$ are entirely determined by the same recursion relations (on the degree) as the expectation values of invariant polynomials of two real antisymmetric matrices. This completes the proof of Theorem 3.1.

10.3 Symplectic Case

The procedure to obtain the loop equations for the real quaternionic antiselfdual two-matrix integral and for the $\tilde{J}$-antisymmetric complex matrix integral and to prove Theorem 3.2 is completely analogous to the one above.

Appendix 3: Calculation of Jacobians

In this appendix we are going to detail the main steps for the computation of the Jacobians (4.3), (4.12), (5.4) and (5.9). For this purpose we will need one of the special limiting cases of the Selberg integral called the Laguerre limit (see for example [19])

$$\begin{aligned} I(\alpha,\gamma,n) &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Bigg(\prod_{1\le i<j\le n}(x_j^2-x_i^2)\Bigg)^{2\gamma}\Bigg(\prod_{k=1}^{n}x_k^{2\alpha-1}\mathrm{e}^{-x_k^2}\Bigg)\mathrm{d}x_1\ldots\mathrm{d}x_n \\ &= \prod_{j=0}^{n-1}\frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+j\gamma)}{\Gamma(1+\gamma)}. \end{aligned} \tag{11.1}$$

The two values of this integral we need are

$$I\left(\frac{1}{2},1,m\right) = m!\frac{(\sqrt{\pi})^m}{2^{m(m-1)}}\prod_{j=1}^{m-1}(2j)!, \tag{11.2}$$

$$I\left(\frac{3}{2},1,m\right) = m!\frac{(\sqrt{\pi})^m}{2^{m^2}}\prod_{j=1}^{m}(2j-1)!. \tag{11.3}$$

The procedure is essentially the same for the four cases. Let us show in detail the first one, Jac_n^O.

The following relation holds true by the block-diagonal decomposition shown in Sect. 4.1

$$\begin{aligned} &\int_{\mathcal{A}_n}\mathrm{d}A\mathrm{e}^{\mathrm{Tr}(\frac{A^2}{2})} \\ &= \mathrm{Jac}_n^O\int_{-\infty}^{\infty}\prod_{1\le i<j\le m}(x_j^2-x_i^2)^2\begin{cases}\prod_{k=1}^m \mathrm{e}^{-x_k^2}\mathrm{d}x_k & \text{if } n=2m,\\ \prod_{k=1}^m x_k^2\mathrm{e}^{-x_k^2}\mathrm{d}x_k & \text{if } n=2m+1\end{cases} \\ &= \mathrm{Jac}_n^O\begin{cases} I(\frac{1}{2},1,m) & \text{if } n=2m,\\ I(\frac{3}{2},1,m) & \text{if } n=2m+1.\end{cases} \end{aligned} \tag{11.4}$$

Computing the Gaussian integral on the left hand side we find

$$\mathrm{Jac}_n^O = (\sqrt{\pi})^{\frac{n(n-1)}{2}}\begin{cases}(I(\frac{1}{2},1,m))^{-1} & \text{if } n=2m,\\ (I(\frac{3}{2},1,m))^{-1} & \text{if } n=2m+1\end{cases}$$

which gives exactly the expression in (4.3).

The Jacobian Jac_{2m}^{Sp} is computed with the same technique from a real quaternionic anti-selfdual Gaussian integral,

$$\left(\frac{1}{2}\right)^{m(m-1)}(\sqrt{\pi})^{m(2m+1)} = \int_{Q\mathcal{A}_m} \mathrm{d}Q\mathrm{e}^{\mathrm{Tr}_0(\frac{Q^2}{2})} = \mathrm{Jac}_{2m}^{Sp} I\left(\frac{3}{2}, 1, m\right). \tag{11.5}$$

In order to compute the two remaining Jacobians $\mathrm{Jac}_n^{U^J}$ and $\mathrm{Jac}_{2m}^{U^{\tilde{J}}}$ we need to introduce two new matrix ensembles. Consider the *Hermitean* $J/\tilde{J}$-antisymmetric one-matrix model. By the $J/\tilde{J}$-antisymmetry of these matrices we know that they can be put into a triangular form by a twisted orthogonal or symplectic matrix respectively. By hermiticity we conclude that the triangular part of this Schur form will be zero, and the diagonal part (eigenvalues) is real. It is easy to argue that the Jacobians for these transformations have to be the same as the ones we seek in Sects. 4.2 and 5.2. This allows us to write the following

$$\begin{aligned}\int \mathrm{d}H\mathrm{e}^{-\mathrm{Tr}(\frac{H^2}{2})} &= \begin{cases}(\sqrt{\pi})^{\frac{n(n-1)}{2}} 2^{m-m^2} & \text{if } \mathcal{J} = J,\\ (\sqrt{\pi})^{2m^2+m} 2^{m-m^2} & \text{if } \mathcal{J} = \tilde{J}\end{cases}\\ &= \mathrm{Jac}_n^{U^{\mathcal{J}}} \begin{cases} I(\frac{1}{2}, 1, m) & \text{if } \mathcal{J} = J \text{ and } n = 2m,\\ I(\frac{3}{2}, 1, m) & \text{if } \mathcal{J} = J \text{ and } n = 2m+1,\\ 2^{2m} I(\frac{3}{2}, 1, m) & \text{if } \mathcal{J} = \tilde{J} \text{ and } n = 2m\end{cases}\end{aligned} \tag{11.6}$$

which gives (4.12) and (5.10).

Appendix 4: Proof of Theorem 6.1

In this appendix we use the graphical representation of the basis of correlation functions introduced in Sect. 6.3.1 to prove Theorem 6.1. The idea is to identify all possible occurrences of elements of the first row and last column of the T matrix (and vice versa for $T^\dagger$) by means of the decomposition of (6.6); then to use the constraints coming from (i) the triangular structure of these matrices, (ii) the contractions of indices within traces, (iii) the propagators (6.9) and (6.10), to represent the result of the integration in a graphical way, leading to the recursion formulae.

12.1 Last column/first row integration

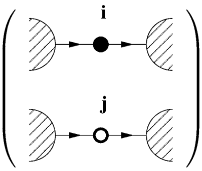
Take the functions defined in 6.3.1 and their graphical representation. We first rewrite (6.6) in a slightly reshuffled form

$$\begin{aligned}&\left(\frac{1}{x-(X+T)}\right)_{i,j}\\ &= \delta_{i,1}\delta_{j,n}\frac{1}{x-\alpha}T_{1,n}\frac{1}{x+\alpha}\\ &\quad + \delta_{i,1}\delta_{j,1}\frac{1}{x-\alpha} + \delta_{i,n}\delta_{j,n}\frac{1}{x+\alpha}\\ &\quad + \left\{\delta_{i,1}(1-\delta_{j,1}-\delta_{j,n})\frac{1}{x-\alpha}\left[\sum_{k=2}^{j} T_{1,k}\left(\frac{1}{x-(\tilde{X}+\tilde{T})}\right)_{k,j}\right]\right.\end{aligned}$$

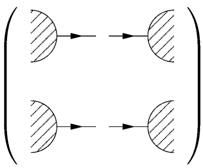
$$
\begin{aligned}
&+(1-\delta_{i,1}-\delta_{i,n})\delta_{j,n}\left[\sum_{l=i}^{n-1}\left(\frac{1}{x-(\tilde{X}+\tilde{T})}\right)_{i,l}T_{l,n}\right]\frac{1}{x+\alpha}\\
&+\delta_{i,1}\delta_{j,n}\frac{1}{x-\alpha}\left[\sum_{\substack{k<l\\=2}}^{n-1}T_{1,k}\left(\frac{1}{x-(\tilde{X}-\tilde{T})}\right)_{k,l}T_{l,n}\right]\frac{1}{x+\alpha}\Bigg\}\\
&+(1-\delta_{i,1}-\delta_{i,n})(1-\delta_{j,1}-\delta_{j,n})\left(\frac{1}{x-(\tilde{X}+\tilde{T})}\right)_{i,j} \qquad (12.1)
\end{aligned}
$$

where α and β are the first eigenvalues of X and Y respectively. We substitute this expression for each resolvent in the integrand, and perform all possible "contractions" of the $T_{1,k}$, $T_{1,n}$ and $T_{k,n}$ variables by means of the propagators (6.9). This can be represented as operations on the diagram associated to the given function. Note that the terms in the last four lines of (12.1) still contain a resolvent (of size $n-2$), while those on the first two lines do not. Let us now enumerate the operations corresponding to each term in the expansion equation (12.1):

• *Operation 1:* The term on the first line, which singles out one T_{1n} variable, removes one resolvent from the integrand, which is represented by erasing a dot in the diagram. Since $T_{1,n}$ can only be contracted with $T^\dagger_{n,1}$, the appearance of this term forces the erasing of a dot of the opposite color, by another application of Operation 1 on a y-type resolvent, somewhere in the diagram. Since this T_{1n} appears in a trace, its left and right neighboring resolvents must have a $T^\dagger$ with one matching index 1 or n. The operation of erasing dots leaves pairs of *free links* with only one dot at their end, carrying such a $T^\dagger$ variable; their role and their weight will be reconsidered in *Operation 3*. The same applies to the other erased dot. Let us now perform the contraction of the selected $T_{1,n}T^\dagger_{n,1}$ pair, giving a factor $(1+b)$. The graphical representation is,

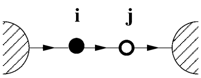


$$\Rightarrow \quad (1+b)\frac{1}{x_i^2-\alpha^2}\frac{1}{y_j^2-\beta^2}$$

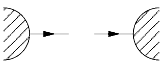


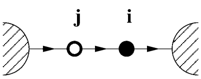
These pairings have to be performed in all inequivalent ways.

• *Operation 2:* The two terms in the second line play a similar role. They also remove a resolvent, which is again represented by erasing a dot. This forces one of the neighbors to be replaced by a similar term. This will be represented by the operation of erasing a link and its two adjacent dots. The possible configurations are

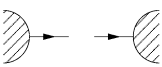


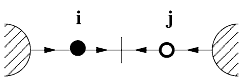
$$\Rightarrow \quad \frac{1}{x_i-\alpha}\frac{1}{y_j-\beta}$$



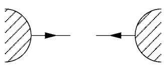


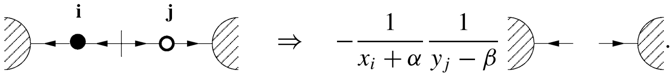
$$\Rightarrow \quad \frac{1}{x_i+\alpha}\frac{1}{y_j+\beta}$$





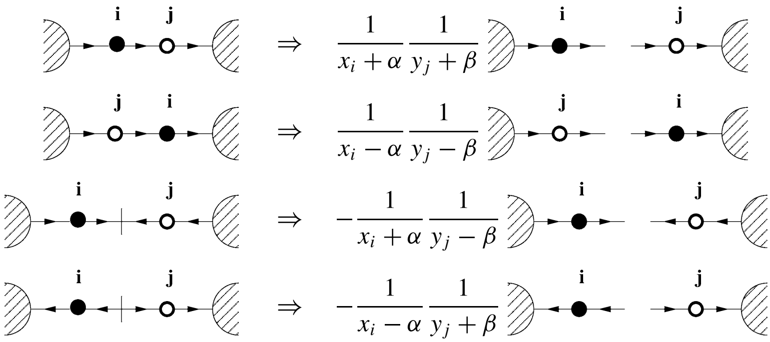
$$\Rightarrow \quad -\frac{1}{x_i-\alpha}\frac{1}{y_j+\beta}$$



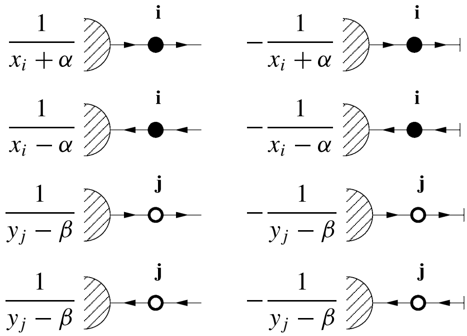


The signs in the last two equations come from the twist of the erased link. This operation also leaves some free links.

• *Operation 3:* All the remaining terms do not remove resolvents so they do not erase any dot in the diagram. Instead, they represent cuts in the links, since each $T_{1,k}$ and $T_{k,n}$ forces a $T^\dagger_{n,l}$ or $T^\dagger_{1,l}$ in a neighbor. We must also consider here all free links created by erasing dots in *Operations 1* and *2*. As discussed above, these terms contain also a T or a $T^\dagger$ variable at the end of the free link, and will contribute to the weight. Graphically we have



for the cutting, and



for the free links coming from *Operations 1* and *2*. In the right column, the bar across the free link indicates the presence of a $\mathcal{J}$ matrix.

• *Operation 4:* Finally, the only term we did not consider (the one with one resolvent and no T variable on the last line of (12.1)) accounts for doing nothing to a dot.

Substituting (12.1) for each resolvent is equivalent to performing Operations 1 to 4 on all dots/links and in all possible ways. After this we have diagrams with free links and missing dots. The final step is to join the remaining free links. This is equivalent to contracting the T and $T^\dagger$ variables in all possible ways. The gluing of free links gives a trivial weight, so this final step is just graphical.

Let us illustrate this procedure on the example treated in Sect. 6.2. The following equation represents the application of Operations 1 to 4 in all possible ways, with their corresponding weights.

$$
\begin{aligned}
\left(\quad + 1 \right) &= 1 + \frac{1}{x-\alpha}\frac{1}{y-\beta} \quad + \frac{1}{x+\alpha}\frac{1}{y+\beta} \\
&\quad + (1+b)\frac{1}{x^2-\alpha^2}\frac{1}{y^2-\beta^2} \quad + \\
&\quad + \frac{1}{x-\alpha}\frac{1}{y-\beta} \quad + \frac{1}{x+\alpha}\frac{1}{y+\beta} \\
&\quad + \frac{1}{x^2-\alpha^2}\frac{1}{y^2-\beta^2}
\end{aligned}
\tag{12.2}
$$

In this equation shaded dots and links represent erased dots and links. Notice that this intermediate formula can be identified term by term with formula (6.12). Gluing of the free links gives

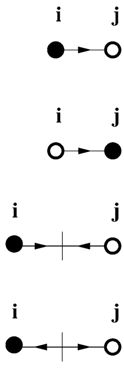
$$
\begin{aligned}
\left(\quad + 1 \right) &= 1 + \frac{1}{x-\alpha}\frac{1}{y-\beta} + \frac{1}{x+\alpha}\frac{1}{y+\beta} + (1+b)\frac{1}{x^2-\alpha^2}\frac{1}{y^2-\beta^2} \\
&\quad \times \left(1 + \frac{1}{x-\alpha}\frac{1}{y-\beta} + \frac{1}{x+\alpha}\frac{1}{y+\beta} \right) \\
&\quad + \frac{1}{x^2-\alpha^2}\frac{1}{y^2-\beta^2}\left(\quad + \quad \right) \\
&= \left(1 + \frac{1}{x-\alpha}\frac{1}{y-\beta} \right)\left(1 + \frac{1}{x+\alpha}\frac{1}{y+\beta} \right)\left(\quad + 1 \right) \\
&\quad + \frac{1}{x^2-\alpha^2}\frac{1}{y^2-\beta^2}\left(\quad + b \right)
\end{aligned}
$$

which matches exactly the result (6.13) in Sect. 6.2 but calculated here using the graphical rules we have defined.

12.2 Computation of the Weight for the Final Diagrams

Consider a reduced problem where no erasing of dots is allowed, i.e. only *Operation 3* and *4* are taken into account. In this case only cutting and gluing is allowed and no difference at the graphical level appears between the J-antisymmetric and the $\tilde{J}$-antisymmetric cases.

To each link in the final diagram G' is attached a weight coming from the different ways we obtain it from the original diagram G. That is, when a link in G' is part of G, we can either cut that original link and glue it again, or just do nothing. Instead, if the link in G' does not belong to G, the only way to obtain it is by gluing cut links. Both contributions will add up to



$$\in G \to \begin{cases} (1 + \frac{1}{x_i+\alpha}\frac{1}{y_j+\beta}) & \text{if the link } \in G', \\ \frac{1}{x_i+\alpha}\frac{1}{y_j+\beta} & \text{if the link } \notin G', \end{cases}$$

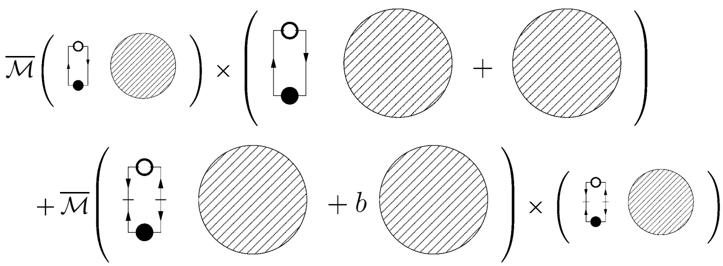
$$\in G \to \begin{cases} (1 + \frac{1}{x_i-\alpha}\frac{1}{y_j-\beta}) & \text{if the link } \in G', \\ \frac{1}{x_i-\alpha}\frac{1}{y_j-\beta} & \text{if the link } \notin G', \end{cases}$$

$$\in G \to \begin{cases} (1 - \frac{1}{x_i+\alpha}\frac{1}{y_j-\beta}) & \text{if the link } \in G', \\ -\frac{1}{x_i+\alpha}\frac{1}{y_j-\beta} & \text{if the link } \notin G', \end{cases}$$

$$\in G \to \begin{cases} (1 - \frac{1}{x_i-\alpha}\frac{1}{y_j+\beta}) & \text{if the link } \in G', \\ -\frac{1}{x_i-\alpha}\frac{1}{y_j+\beta} & \text{if the link } \notin G'. \end{cases}$$

Returning now to the original complete problem, where the erasing of dots is allowed, one notices that the weight found in Operations 1 and/or 2 by erasing a pair of dots is the same as the one obtained by forming a cycle with this pair of dots with Operations 3 and 4. Indeed, consider a minimal cycle (i.e. a cycle of length 2) in G' and erase from G the dots in this minimal cycle using *Operation 1* and if possible *Operation 2*. *Operation 1* contributes a factor $(1+b)$; if the minimal cycle in G' is non twisted, we assign it the weight 1, while if it is, we assign it the weight b.[4] Iterating this operation for all minimal cycles in G', one finds that the two procedures, erasing pairs of dots or forming minimal cycles with the same dots, produce the same weight. A slightly different manipulation is needed when the minimal cycle is present already in G. The outcome will be the same.

Here is an example of the kind of diagrams which have the same weight within the complete problem:



where $\overline{\mathcal{M}}(G') \equiv \overline{\mathcal{M}}_G^{G'}$ is the weight associated with G', for G the original diagram.

We proved that for the complete problem, the recursion matrix is the same as that of the reduced problem. Reassembling everything together and using back the sets of signs s and s', we get the recursion matrix

$$\overline{\mathcal{M}}_G^{G'} = \Bigg(\prod_{((x_i,s(i)),(x_{\pi(i)},s'(\pi(i))))\in G'} \left(1 + \frac{1}{s(i)x_i+\alpha}\frac{1}{s'(\pi(i))y_{\pi(i)}+\beta}\right)$$

[4]This is the origin of the additive ± 1's coming with every minimal cycle in the basis of functions equation (6.18).

$$\times \prod_{((x_i,s(i)),(y_{\pi(i)},s'(\pi(i))))\notin G'} \left(\frac{1}{s(i)x_i+\alpha}\frac{1}{s'(\pi(i))y_{\pi(i)}+\beta}\right)\Bigg)$$
$$\times \Bigg(\prod_{((x_i,s(i)),(y_{\pi'(i)},s'(\pi'(i))))\in G'} \left(1+\frac{1}{s(i)x_i-\alpha}\frac{1}{s'(\pi'(i))y_{\pi'(i)}-\beta}\right)$$
$$\times \prod_{((x_i,s(i)),(y_{\pi'(i)},s'(\pi'(i))))\notin G'} \left(\frac{1}{s(i)x_i-\alpha}\frac{1}{s'(\pi'(i))y_{\pi'(i)}-\beta}\right)\Bigg).$$

Labeling each diagram using the labels in (6.18), (6.19) and (6.22) we get

$$\begin{aligned}
&\overline{\mathcal{M}}^{\{(\tau,t),(\tau',t')\}}_{\{(\pi,s),(\pi',s')\}} \\
&= \left(\prod_{i=1}^{R}\left(\delta_{\pi(i),\tau(i)}\delta_{s(i),t(i)}\delta_{s'(\pi(i)),t'(\pi(i))}+\frac{1}{s(i)x_i+\alpha}\frac{1}{s'(\pi(i))y_{\pi(i)}+\beta}\right)\right) \\
&\quad\times\left(\prod_{i=1}^{R}\left(\delta_{\pi'(i),\tau'(i)}\delta_{s(i),t(i)}\delta_{s'(\pi'(i)),t'(\pi'(i))}+\frac{1}{s(i)x_i-\alpha}\frac{1}{s'(\pi'(i))y_{\pi'(i)}-\beta}\right)\right)
\end{aligned} \tag{12.3}$$

which completes the proof.

Appendix 5: Relations between Orthogonal/Symplectic and Unitary Recursion Equations

In this appendix we relate tetrads $\omega=\{\sigma,\tau,s,t\}$ introduced in Sect. 6.3.1 and permutations $\pi\in\mathfrak{S}_{2R}$, and more precisely to show the bijection between the set of *equivalence classes* $[\omega]$ and $\mathfrak{S}_{2R}$; this leads to an important relation between the recursion matrix $\overline{\mathcal{M}}$ and the basis of correlation functions $F^{\mathcal{J}}$ for the orthogonal/symplectic case and the recursion matrix $\mathcal{M}$

$$\mathcal{M}^{(2R)}_{\pi,\pi'}(\{x\},\{y\},\alpha,\beta)=\prod_{i=1}^{2R}\left(\delta_{\pi(i),\pi'(i)}+\frac{1}{x_i-\alpha}\frac{1}{y_{\pi(i)}-\beta}\right) \tag{13.1}$$

and the basis of correlation functions F^U

$$F^U_{\pi,\pi'}(\{x\},\{y\},A,B)=\prod_{k=1}^{p}\left(\delta_{R_k,1}+\mathrm{tr}\left(\prod_{l=1}^{R_k}\frac{1}{x_{i_{k,l}}-A}\frac{1}{y_{j_{k,l}}-B}\right)\right) \tag{13.2}$$

found in [12] in the unitary case.

13.1 Bijection between $\mathfrak{S}_{2R}$ and Equivalence Classes in $\mathfrak{S}_R\times\mathfrak{S}_R\times\mathbb{Z}_2^R\times\mathbb{Z}_2^R$

Consider pairs of permutations σ and τ belonging to $\mathfrak{S}_R$, pairs of sets of R signs s and t belonging to $\mathbb{Z}_2^R$. As explained in Sect. 6.3.1, $\sigma\circ\tau^{-1}$ represents a permutation of R (black) points with signs $s(i)$ attached to them, and $\tau\circ\sigma^{-1}$ a permutation of R (white) points with signs $t(i)$. To get one representative of the equivalence class $[\{\sigma,\tau,s,t\}]$, we fix one sign s_i in every cycle of $\sigma\circ\tau^{-1}$ to be $+1$.

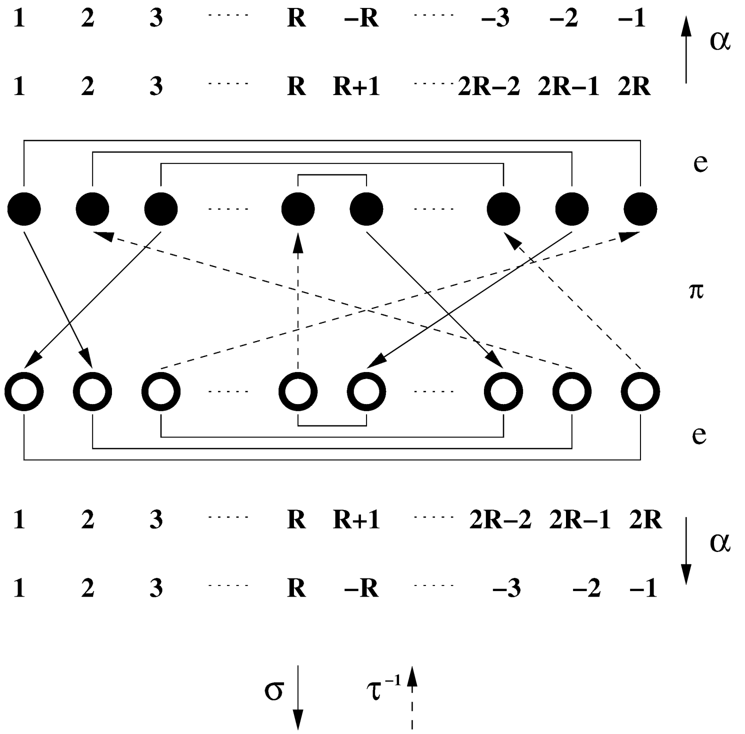


Fig. 1 Representation of the bijection. The set of lines, irrespective of their type (solid or dashed) represents $\pi \in \mathfrak{S}_{2R}$. Solid lines represent σ and dashed ones represent τ. The signs s, resp t, are the signs at the origin, resp. the end, of the solid lines. The arrows, though redundant, are meant to help the reader follow the iterations 1–4 above

Lemma 13.1 *There is a bijection between $\mathfrak{S}_{2R}$ and the equivalence classes of $\mathfrak{S}_R \times \mathfrak{S}_R \times \mathbb{Z}_2^R \times \mathbb{Z}_2^R$.*

Proof To construct the bijection we take a permutation $\pi \in \mathfrak{S}_{2R}$. We will relabel the indices $i = 1, \dots, 2R$ and call them $\alpha(i) = 1, \dots, R, -R, \dots, -1$ (see Fig. 1),

$$\alpha(i) = \begin{cases} i & \text{if } i \le R. \\ i - (2R+1) & \text{if } i > R. \end{cases}$$

Define also the auxiliary "rainbow" permutation $e(i) = 2R - i$ for which $e^2 = \mathrm{id}$ and $e(\alpha(i)) \equiv \alpha(e(i)) = -\alpha(i)$.

With this new labeling we construct the permutations σ and τ and the signs s and t as follows

- 1. Begin with $i = 1$
- 2. Set $s(|\alpha(i)|) = \mathrm{sgn}(\alpha(i))$, then $j := \pi(i)$ and set $\sigma(|\alpha(i)|) = |\alpha(j)|$
- 3. Set $t(|\alpha(j)|) = \mathrm{sgn}(\alpha(j))$, then $k = e\pi^{-1}e(j)$ and set $\tau^{-1}(|\alpha(j)|) = |\alpha(k)|$
- 4. If we do not close a cycle set $i = k$ and go back to 2.

When we close a cycle of $\sigma \circ \tau^{-1}$, (for example, to close the first cycle we must find again $k = 1$ at the end of step 3.), we must open a new cycle. To do so, look at which positive

α-type indices we have not used yet and choose the smallest one. Set i equal to this value and restart from point 2. When there is no positive index left, the last cycle is completed and the tetrad $\{\sigma, \tau, s, t\}$ is constructed. $\square$

This procedure is illustrated in Fig. 1: there, the permutation in $\mathfrak{S}_{2R}$ is $\pi = (1, 2, R + 2, 2R, R + 3, R + 1, 3)(R)(\cdots)$. Following the above rules we determine from it $\sigma = (1, 2, R, 3)(\cdots)$, $\tau = (1, 3)(2)(R)(\cdots)$, $s = (+, -, +, \dots, -)$ and $t = (+, +, -, \dots, -)$.

Note that the first s sign of every cycle is positive by construction. Note also that conversely, constructing π from the tetrad $\{\sigma, \tau, s, t\}$ can be done with the same kind of procedure: this is clearly seen in the example. By construction, in this reverse operation, π depends only on the equivalence class $[\omega]$. Since the method is deterministic in both directions we have a bijection.

13.2 Relation between $\overline{\mathcal{M}}^{(R)}$ and $\mathcal{M}^{(2R)}$

Consider now the set of variables

$$\begin{aligned}
&\{x\}_{2R} = \{x_1, \dots, x_R, x_{R+1} = -x_R, \dots, x_{2R} = -x_1\},\\
&\{x\}_R = \{x_1, \dots, x_R\},\\
&\{y\}_{2R} = \{y_1, \dots, y_R, y_{R+1} = -y_R, \dots, y_{2R} = -y_1\},\\
&\{y\}_R = \{y_1, \dots, y_R\}
\end{aligned}$$

and consider also the pairs of indices

$$\begin{aligned}
&\pi, \pi' \in \mathfrak{S}_{2R},\\
&[\{\sigma, \sigma', s, s'\}], [\{\tau, \tau', t, t'\}]\\
&\quad \in \text{Equivalence Classes of } \mathfrak{S}_R \times \mathfrak{S}_R \times \mathbb{Z}_2^R \times \mathbb{Z}_2^R
\end{aligned} \tag{13.3}$$

where the indices are related through the bijection shown above. Using these definitions it is easy to verify that

$$\begin{aligned}
&\overline{\mathcal{M}}^{(R)\{\tau,\tau',t,t'\}}_{\{\sigma,\sigma',s,s'\}}(\{x\}_R, \{y\}_R, \alpha, \beta)\\
&\quad = \left(\prod_{i=1}^{R}\left(\delta_{\sigma(i)\tau(i)}\delta_{s(i)t(i)}\delta_{s'(\sigma(i))t'(\sigma(i))} + \frac{1}{s(i)x_i + \alpha}\frac{1}{s'(\sigma(i))y_{\sigma(i)} + \beta}\right)\right)\\
&\qquad \times \left(\prod_{i=1}^{R}\left(\delta_{\sigma'(i)\tau'(i)}\delta_{s(i)t(i)}\delta_{s'(\sigma'(i))t'(\sigma'(i))} + \frac{1}{s(i)x_i - \alpha}\frac{1}{s'(\sigma'(i))y_{\sigma'(i)} - \beta}\right)\right)\\
&\quad = \left(\prod_{i=1}^{2R}\left(\delta_{\pi(i),\pi'(i)} + \frac{1}{x_i + \alpha}\frac{1}{y_{\pi(i)} + \beta}\right)\right)\\
&\quad = \mathcal{M}^{(2R)}_{\pi,\pi'}(\{x\}_{2R}, \{y\}_{2R}, -\alpha, -\beta)
\end{aligned}$$

i.e. the two matrices encountered in the orthogonal/symplectic and unitary cases are in fact the same. In this calculation, we have used the bijection to reexpress the Kronecker δ symbols:

$$\delta_{\sigma(i),\tau(i)}\delta_{s(i),t(i)}\delta_{s'(\sigma(i)),t'(\sigma(i))} = \delta_{\pi(\alpha^{-1}(s(i)i)),\pi'(\alpha^{-1}(s(i)i))},$$

$$\delta_{\sigma'(i)\tau'(i)}\delta_{s(i)t(i)}\delta_{s'(\sigma'(i))t'(\sigma'(i))} = \delta_{\pi(\alpha^{-1}(-s(i)i)),\pi'(\alpha^{-1}(-s(i)i))}$$

and to relate the $\{x\}_R$ and $\{y\}_R$ variables with the $\{x\}_{2R}$ and $\{y\}_{2R}$ variables according to

$$s(i)x_i = x_{\alpha^{-1}(s(i)i)}, \qquad s'(\sigma(i))y_{\sigma(i)} = y_{\pi(\alpha^{-1}(s(i)i))},$$
$$-s'(\sigma'(i))y_{\sigma'(i)} = y_{\pi(\alpha^{-1}(-s(i)i))}.$$

13.3 Relation between $F^{\mathcal{J}}$ and F^U

Finally consider the basis of correlation functions in the unitary case for $2R$ X-type and $2R$ Y-type resolvents. In particular consider the components $F^U_{\pi,e\pi e}(\{x\}_{2R},\{y\}_{2R},A,B)$. Call $[\omega]=[\{\sigma,\tau,s,t\}]$ the equivalence class of tetrads corresponding to π.

The function $F^U_{\pi,e\pi e}(\{x\}_{2R},\{y\}_{2R},A,B)$ can be constructed from the kind of diagrams shown in Fig. 1 by following the $\pi e\pi^{-1}e$ cycles. These cycles are, by construction, the same ones we follow with $\sigma\tau^{-1}$, with the only difference that π is a permutation of $2R$ elements instead of R. Because of this, $\pi e\pi^{-1}e$ contains two different representatives of the equivalence class $[\omega]$, and since the functions $F^{\mathcal{J}}$ are independent of the class representative, F^U contains twice the same function $F^{\mathcal{J}}$. We can write this as

$$F^U_{\pi,e\pi e}(\{x\}_{2R},\{y\}_{2R},A,B) = (F^{\mathcal{J}}_{\omega}(\{x\}_R,\{y\}_R,A,B))^2. \qquad (13.4)$$

The sign in (6.22) is, however, not easy to read from π and e.

References

1. Di Francesco, P., Ginsparg, P., Zinn-Justin, J.: 2D gravity and random matrices. Phys. Rep. **254**, 1–133 (1995)
2. Eynard, B.: Random matrices. http://www-spht.cea.fr/articles_k2/t01/014/publi.pdf (2000)
3. Guhr, T., Mueller-Groeling, A., Weidenmueller, H.A.: Phys. Rep. **299**, 189–425 (1998), cond-mat/9707301
4. Forrester, P.J., Snaith, N.C., Verbaarschot, J.J.M.: J. Phys. A **36**, R1–R10 [Special issue: Random Matrix Theory], cond-mat/0303207
5. Harish-Chandra: Am. J. Math. **79**, 87–120 (1957)
6. Itzykson, C., Zuber, J.-B.: J. Math. Phys. **21**, 411–421 (1980)
7. Zinn-Justin, P., Zuber, J.-B.: J. Phys. A **36**, 3173–3193 (2003) [Special issue: Forrester, P.J., Snaith, N.C., Verbaarschot, J.J.M. (eds.), Random Matrix Theory], math-ph/0209019
8. Kogan, I.I., Morozov, A., Semenoff, G.W., Weiss, N.: Nucl. Phys. B **395**, 547–580 (1993), hep-th/9208012
9. Morozov, A.: Mod. Phys. Lett. A **7**, 3503–3508 (1992), hep-th/9209074
10. Shatashvili, S.: Commun. Math. Phys. **154**, 421–432 (1993), hep-th/9209083
11. Eynard, B.: A short note about Morozov's formula. math-ph/0406063
12. Eynard, B., Prats Ferrer, A.: Commun. Math. Phys. **264**, 115–144 (2005), hep-th/0502041; Erratum to be published
13. Duistermaat, J.J., Heckman, G.J.: Invent. Math. **69**, 259–268 (1982)
14. Duistermaat, J.J., Heckman, G.J.: Invent. Math. **72**, 153–158 (1983)
15. Szabo, R.J.: Equivariant localization of path integrals. hep-th/9608068
16. Karshon, Y.: Lecture notes on group actions on manifolds. http://www.ma.huji.ac.il/~karshon/teaching/1996-97/actions/lecture-notes.html (1996–1997)
17. Stone, M.: Nucl. Phys. B **314**, 557–586 (1989)
18. Bump, D.: Lie Groups. Graduate Texts in Mathematics, vol. 225. Springer, Berlin (2004)
19. Mehta, M.L.: Random Matrices. Academic Press, New York (1991)
20. Ginibre, J.: J. Math. Phys. **6**, 440–449 (1965)
21. Knapp, A.W.: Representation Theory of Semi-simple Groups. Princeton University Press, Princeton (1986)

22. Fulton, W., Harris, J.: Representation Theory. Graduate Texts in Mathematics, vol. 129. Springer, Berlin (1991)
23. Efetov, K.B.: Supersymmetry in Disorder and Chaos. Cambridge University Press, Cambridge (1999)
24. Di Francesco, P., Zinn-Justin, P.: From orbital varieties to alternating sign matrices. math-ph/0512047, to appear in the proceedings of FPSAC'06
25. Fyodorov, Y.V., Khoruzhenko, B.A.: On absolute moments of characteristic polynomials of a certain class of complex random matrices. math-ph/0602032
26. Fyodorov, Y.V.: On permanental polynomials of certain random matrices. math-ph/0602039
27. Brézin, É., Hikami, S.: An extension of the Harish-Chandra–Itzykson–Zuber integral. math-ph/0208002